A MULTIOBJECTIVE OPTIMIZATION APPROACH FOR THE PERFORMANCE MEASURE OF STATISTICAL PROCESS CONTROL

BY

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DEDICATION

This work is dedicated to Allah, the mighty, the bestower of wisdom and understanding, and to my late parents Late Chief Imam, Alhaji Jamiu Ajibade Abdul-Salam and Late Mrs. Ajibade Rafat Aderinlola. May Allah forgive them and have mercy on them. (Amin).

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LIST OF ABBREVIATIONS

OP	Optimization Problem
MOP	Multiobjective Optimization Problem
MO	Multiobjective Optimization
KKT	Karush Kuhn Tucker
ARL	Average Run Length
EWMA	Exponentially Weighted Moving Average
AR(1)	Autoregressive process of order one
CUSUM	Cumulative Sum
CSEWMA	Combined Shewhart EWMA Control Chart
FIR	Fast Initial Response
MFIR	Modified Fast Initial Response
GFIR	Generalized Fast Initial Response
EA	Evolutionary Algorithm
GA	Genetic Algorithm

THESIS ABSTRACT

NAME:	Ajibade Ganiyu Ayodele
TITLE OF STUDY:	A Multiobjective Optimization Approach for the Performance Measure of Statistical Process Control
MAJOR FIELD:	Mathematics
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In this Thesis work, Multiobjective optimization design for statistical process control is proposed. Some adjustments are also made to some existing control schemes. These further enhance their performances.

أطروحة مجردة

في هذا العمل أطروحة، ويقترح موضوعي متعدد التصميم الأمثل موضوعي ل مراقبة العملية الإحصائية. يتم إجراء بعض التعديلات أيضا على بعض مخططات الرقابة القائمة. هذه زيادة تعزيز أدائهم.

CHAPTER 1

INTRODUCTION

Control chart is a statistical tool used in monitoring processes. It is very powerful because of its ability to detect variations in a given process. The concept of control chart was first suggested by Walter A. Shewhart in the 1920s. The most popular control charts are Shewhart $\bar{X}$ -chart, Cumulative Sum control chart (CUSUM) introduced by Page[24] and Exponentially Weighted Moving Average (EWMA) control chart introduced by Roberts[25].

EWMA is one of the advanced techniques used in monitoring statistical processes. Due to its high sensitivity of detecting small changes in a process mean or standard deviation, it is widely used in the chemical industries where little process disturbances often result to a huge financial penalties[21].

While CUSUM and EWMA are good in detecting small shifts in a given process, Shewhart $\bar{X}$ -chart is effective in detecting big shifts in a given process. The combined EWMA and Shewhart $\bar{X}$ -chart are often used to monitor a process where all the mean shifts regions are to be effectively detected. Whereas, the mean shifts in a process can either be small, moderate or large.

Consequently, to accommodate all different sizes of mean shifts in a process, Lucas and Saccucci[32, 33] suggested a combined Shewhart-EWMA (CSEWMA) control chart. Similarly, Woodall and Maragah[34] stressed the need to always combine a Shewhart chart with an EWMA chart.

The performance assessment of a control chart is usually by the average run length (ARL). The ARL of a charting procedure is the number of samples to be taken before a false alarm is observed in the process. The performance can be evaluated by two values ARL_0 and ARL_1. ARL_0 is the expected number of samples before an out-of-control false alarm is detected when the process is at in-control state while ARL_1 is the expected number of samples before an out-of-control false alarm occurs when the process is shifted to an out-of-control state. A chart is considered to be more effective than other chart if it has a smaller ARL_1 (out-of-control ARL) values at more points Wu et al[39].

In a process control, there are mainly three mean shifts regions namely; small, moderate and large mean shifts regions. The ARL over the three regions and on the same decision parameters are to be minimized simultaneously. This leads to conflicting objectives thereby resulting to Multiobjective optimization problems. In this thesis, We formulate this concept as a Multiobjective optimization problems.

The Karush Kuhn Tucker (KKT) first order optimality conditions is given and the Pareto optimal points are obtained. min-EWMA control scheme is proposed.

We equally suggest the generalized time varying initial response for control charts. Lastly, we provide some equivalent ways for which the generalized control charts can be written. The comparison of our control procedures with other control schemes mention in this article, on this topic, reveal that our designs performed better.

Organization of the thesis is as follows: In Chapter 2, we gave the literature review on MOP and control charts. In Chapter 3, we discussed the concepts of optimization and the optimality notions on MOP. In Chapter 4, we discussed control charts and proposed some enhancement techniques to the existing control schemes. In Chapter 5, we gave the optimization design proposed in this Thesis work. Chapter 6 contains the conclusive remarks of the work.

CHAPTER 2

LITERATURE REVIEWS

2.1 Literature review on MOP

Multiobjective optimization problem (MOP) is an Optimization problem (OP) with more than one objective. It can either be Linear, Non-linear or the combination of both. Information about Linear OP can be found in [5] and Non-linear OP in [4, 9]. The optimality notions on MOP can be found in [1, 3].

The purpose of MO is to simultaneously optimize multiple conflicting objectives. Due to the conflicting nature of the objectives, a feasible solution optimizing all the objectives at the same time does not exist. Consequently, the goal of MOP is to obtain the nondominated set. For a MOP, a nondominated point in the objective space coincide to an efficient solution in decision space while the efficient solution is defined as a solution for which an improvement in one objective will continually worsening at least one of the other objectives. So, the set of all nondominated points forms the nondominated set in objective space.

The classical means of solving such problems were mainly focused on scalarizing multiple objectives into a single objective and solve[1, 9, 8]. Whereas the Evolutionary means have been to solve a MOP as it is. It is a stochastic optimization methods that mimics the process of natural evolution[15, 13, 12, 14, 42]. It operates on two basic principles namely variation and selection. Deb[13] gave the post optimality conditions for which solutions obtained from evolutionary algorithms (EA) alone or a hybrid EA-local search combination can be very close to being KKT points.

2.2 Literature review on Control Charts

Control chart is a statistical tool used in monitoring processes. It is very powerful because of its ability to detect variations in a given process. The concept of control chart was first suggested by Walter A. Shewhart in the 1920s. The most popular control charts are Shewhart $\bar{X}$ -chart, Cumulative sum chart (CUSUM), exponentially weighted moving average control chart. While CUSUM and EWMA are good in detecting small shifts in a given process, Shewhart $\bar{X}$ -chart is effective in detecting big shifts in a given process. The combined EWMA and Shewhart $\bar{X}$ -chart are often used to monitor a process where all the mean shifts regions are to be effectively detected. Whereas, the mean shifts in a process can either be small, moderate or large.

Consequently, to accommodate all different sizes of shifts in a process, Lucas and Saccucci[32, 33] suggested a combined Shewhart-EWMA (CSEWMA) control chart. Similarly, Woodall and Maragah[34] stressed the need to always combine a Shewhart $\bar{X}$ -chart with an EWMA chart.

Roberts[25] introduced Exponentially Weighted Moving Average(EWMA) Control Chart which is one of the advanced techniques used in monitoring statistical processes. Due to the high sensitivity of detecting small changes in a process mean or standard deviation, EWMA is widely used in the chemical industries where small process disturbances often result to serious financial penalties[21].

ARL is the most used method of evaluating the performance of any control chart.The methods to evaluate the performance of EWMA control charts for serially correlated have been studied in [20]. They used simulation method based on the presence of autocorrelation for EWMA control chart. The ARL and steady state ARL of EWMA were estimated numerically in [35] using an integral equation approach and a Markov chain approach to investigate EWMA and CUSUM procedures for the process mean when data resemble an AR(1) process with additional random error term. The explicit formula for ARL values of an in-control and out of control process for EWMA control chart with an exponential white noise and for trend stationary exponential was given in [26, 29]. The authors used an integral equation approach and derive a Fredholm integral equation of second type for the ARL. Gan [44] studied the ARL for EWMA control charts for the exponential distribution by using differential equations.

Steiner[40] proposed the time varying FIR feature for EWMA control chart. Haq[41] later modified this FIR feature to further enhance the performance of

EWMA scheme. Champ et al.[28] proposed the generalized control procedure for which EWMA,CUSUM and Shewhart charts are special cases.

In this Thesis work, min-EWMA control scheme is proposed for which classical EWMA is a special case. The generalized FIR feature for control charts is suggested for which FIR by Steiner[40] is a special case. The equivalent control schemes are proposed. And lastly, MOP for statistical process control is proposed and the trade off points for the three shift regions are obtained.

CHAPTER 3

OPTIMIZATION STUDY

Optimization is the field of study aimed at finding the best allocation of scarce resources among competitive activities. It is a valuable tool for decision making which aids managers to select the best alternatives out of many possible courses of actions. It finds applications in business, economics, statistics, engineering, physics and medicine to mention a few. When only one objective is involved in an optimization problem (OP), it is called a single objective optimization problem while when there are more than one objective, it is called Multiobjective optimization problem (MOP).

In Multiobjective optimization, there are multiple conflicting objectives. So, improving one objective will deteriorate the value of others, resulting to a compromise between solutions. It is assumed that no single solution will optimize all objectives simultaneously because this would be a trivial case. Multiobjective optimization problems in which there is competition between objectives may have no single or unique optimal solution. Multiobjective optimization problems are also referred to as multicriteria or vector optimization problems. It finds applications in many real world problems which includes; fields of Medicine, Engineering, Physics, Statistics, mining and finance.

Let $X \subseteq \Re^n$ and let $f : X \to \Re^m$, the general Multiobjective optimization problems is stated as follows:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \\ &\qquad\qquad x \in X \end{aligned} \tag{3.1}$$

The constraint set can be of the form

$$X = \{x : h(x) = 0, g(x) \leq 0\}, \tag{3.2}$$

where $\quad h : \Re^n \to \Re^k, g : \Re^n \to \Re^p.$

In general, MOP can take any of these forms:

- Minimize all the objective functions.
- Maximize all the objective functions.
- Minimize some and maximize others.

For whatever form it takes, we can easily convert it to the equivalent form in standard form i.e, minimization problem.

3.1 Basic Concepts of Optimization

In this Section, we shall discuss some theories of Optimization problems, the optimality notions on MOP and the method of solving. Now, we start by making the following definitions of convex sets and functions.

3.1.1 Convex Sets and Convex Functions

Definition 3.1 *A set $X \subseteq \Re^n$ is called convex if for any two points $x, y \in X$, we have*

$$\lambda x + (1 - \lambda) y \in X \tag{3.3}$$

for all $\lambda \in [0, 1]$.

Definition 3.2 *A set $X \subseteq \Re^n$ is called strictly convex if for any two points $x, y \in X, \quad x \neq y$, we have*

$$\lambda x + (1 - \lambda) y \in X \tag{3.4}$$

for all $\lambda \in (0, 1)$.

Definition 3.3 *Let X be a nonempty convex subset of $\Re^n$. A function $f : X \to \Re$ is called convex if*

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \tag{3.5}$$

for all $x, y \in X$ *and all* $\lambda \in [0, 1]$.

Definition 3.4 *Let* X *be a nonempty convex subset of* $\Re^n$. *A function* $f : X \to \Re$ *is called strictly convex if*

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y) \tag{3.6}$$

for all $x, y \in X$, $x \neq y$ *and all* $\lambda \in (0, 1)$.

Definition 3.5 *Let* X *be a nonempty convex subset of* $\Re^n$. *A function* $f : X \to \Re$ *is called concave if* $-f$ *is convex. That is, if*

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y) \tag{3.7}$$

for all $x, y \in X$ *and all* $\lambda \in [0, 1]$.

Definition 3.6 *Let* X *be a nonempty convex subset of* $\Re^n$. *A function* $f : X \to \Re$ *is called quasiconvex if*

$$f(\lambda x + (1 - \lambda)y) \leq \max[f(x),\ f(y)] \tag{3.8}$$

for all $x, y \in X$ *and all* $\lambda \in (0, 1)$.

Definition 3.7 *Let* X *be a nonempty convex subset of* $\Re^n$. *A function* $f : X \to \Re$

is called quasiconcave if

$$f(\lambda x + (1-\lambda)y) \geq \min[f(x),\ f(y)] \tag{3.9}$$

for all $x, y \in X$ and all $\lambda \in (0,1)$.

3.1.2 Characterization of Convexity and Quasiconvexity for Twice Differentiable Functions.

There is an additional criteria to ascertain the convexity and quasiconvexity of twice differentiable functions. These are given in the following Theorems.

Theorem 3.1 *Let X be a nonempty convex subsets of $\Re^n$ and suppose that $f : X \to \Re$ has a continuous second order partial derivatives on an open set containing X. Let $\mathbf{H}$ be the Hessian matrix on X defined by*

$$\mathbf{H}(x) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial^2 x_1} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial^2 x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial^2 x_n} \end{pmatrix}. \tag{3.10}$$

Then f is convex if and only if the Hessian matrix is positive semidefinite at each point in X.

Theorem 3.2 *Let X be a nonempty convex subsets of $\Re^n$ and suppose that $f : X \to \Re$ has a continuous second order partial derivatives on an open set con-*

taining X. The kth-order bordered Hessian matrix $\mathbf{D}_k(f,x)$ of a twice continuously differentiable function f at a point $x \in X$ is defined by

$$\mathbf{D}_k(f,x) = det \begin{pmatrix} 0 & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_k} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial^2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_k} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f}{\partial x_k} & \frac{\partial^2 f}{\partial x_k \partial x_1} & \cdots & \frac{\partial^2 f}{\partial^2 x_k} \end{pmatrix}, k = 1, 2, ...n. \qquad (3.11)$$

(a) *If f is quasiconvex on a solid (non-empty interior) convex set X, then*

$$(-1)^k \mathbf{D}_k(f,x) \leq 0, \quad k = 1, 2, ..., n, \text{ for every } x \in X.$$

(b) *A necessary condition for quasiconvexity (in fact, strict quasiconvexity) is*

$$(-1)^k \mathbf{D}_k(f,x) < 0, \quad k = 1, 2, ..., n, \text{ for every } x \in X.$$

Remark : *We have quasiconcavity if the inequality reverses.*

3.2 Optimality Notions of Multiobjective Optimization Problems

In this section, we shall discuss the optimality notions or concepts of MOP. That is, the minimal or maximal elements of a componentwise inequality between vectors. And the existence of such minimal or maximal elements shall be briefly discussed.

Definition 3.8 *Let K be a nonempty subset of $\Re^n$. The set K is called*

(i) a cone if $x \in K$ and $\lambda \geq 0$, we have $\lambda x \in K$.

(ii) a pointed cone if for any cone K, we have $K \cap -K = 0$.

Lemma 3.1 *A cone $K \subseteq \Re^n$ is convex if and only if $K + K \subseteq K$.*

3.2.1 Partially Ordered Sets

Definition 3.9 *Let X be a real linear space.*

(i) Each nonempty subset $\Re$ of the product space XxX is called a binary relation $\Re$ on X and we write $x\Re y$ for $(x, y) \in \Re$.

(ii) Every binary relation $\leq$ on X is called a partial ordering on X, if the following conditions are satisfied. (for any $u, x, y, v \in X$)

(a) $x \leq x$ (Reflexive)

(b) $x \leq y$, $y \leq v \Rightarrow x \leq v$ (Transitive)

(c) $x \leq y$, $u \leq v \Rightarrow x + u \leq y + v$

(d) $x \leq y$, $\alpha \in \Re_+ \Rightarrow \alpha x \leq \alpha y$

(iii) A partial ordering $\leq$ on X is called antisymmetric if the following condition holds for arbitrary $x, y \in X$,

$x \leq y$, $y \leq x \Rightarrow x = y$.

(iv) A real linear space X is called a partially ordered linear space if it is equipped with partial ordering.

We shall now state the Theorem that paves way for the comparison of two arbitrary elements in $\Re^n$.

Theorem 3.3 *Let X be a real linear space. Let $\leq$ be a partial ordering on X,*

(i) then the set

$$K := \{x \in X \mid 0_X \leq x\}$$

is a convex cone and in case $\leq$ is antisymmetric, then K is pointed.

(ii) If $K \subseteq \Re^n$ is a convex cone, then the binary relation

$$\leq_K := \{y - x \in K \mid (x, y) \in X\}$$

is a partial ordering on X and in case K is pointed, then $\leq$ is antisymmetric.

Definition 3.10 *A convex cone K equipped with the partial ordering on $\Re^n$ is called an ordering cone or positive cone.*

3.2.2 Existence of a solution

In this subsection, we shall state some important results in form of definitions or Theorems that guarantee the existence of a solution of any OP.

Definition 3.11 *Let $X \subseteq \Re^n$ and let $f : X \to \Re$. A point $\bar{x} \in X$ with $f(\bar{x}) < \infty$ is called the local minimum of f relative to X if there is $\epsilon > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in \mathcal{B}(\bar{x}; \epsilon) \cap X$.*

Definition 3.12 *Let $X \subseteq \Re^n$ and let $f : X \to \Re$. A point $\bar{x} \in X$ with $f(\bar{x}) < \infty$*

is called the local maximum of f relative to X if there is $\epsilon > 0$ such that $f(\bar{x}) \geq f(x)$ for all $x \in \mathcal{B}(\bar{x};\epsilon) \cap X$.

Remark : *We say that f has a global or absolute minimum at $\bar{x}$ relative to X if $f(\bar{x}) \leq f(x)$ for all $x \in X$.*

Theorem 3.4 ***Weierstrass existence theorem:*** *Let X be a nonempty compact subset of $\Re^n$ and let $f : X \to \Re$ be a continuous function. Then, there exist the point $\bar{x} \in X$ and the point $\bar{y} \in X$ such that*

$$f(\bar{x}) = \inf\{f(x) \mid x \in X\} \text{ and } f(\bar{y}) = \sup\{f(x) \mid x \in X\}.$$

The notions on the optimal elements of a nonempty subset of a partially ordered set in $\Re^n$ actually mean the minimal or maximal elements. However, in some cases, these concepts need further classifications. For example, to know whether it is strongly minimal or weakly minimal (strongly maximal or weakly maximal) elements. Our target is to present the characterizations of these concepts in this section.

Definition 3.13 *Let X be a nonempty subset of a partially ordered linear space with an ordering cone K*

(i) An element $\bar{x} \in X$ is called minimal element of the set X if

$$(\{\bar{x}\} - K) \cap X \subseteq \{\bar{x}\} + K \tag{3.12}$$

(ii) An element $\bar{x} \in X$ is called a maximal element of the set X if

$$(\{\bar{x}\} + K) \cap X \subseteq \{\bar{x}\} - K \tag{3.13}$$

Remarks : *Minimal elements of a set are often referred to as the efficient solutions or Pareto optimal points or nondominated points.*

Remark : If K is a pointed ordering cone, then equations (3.12) and (3.13) can be replaced by

$$(\{\bar{x}\} - K) \cap X = \{\bar{x}\}\,,\ i.e\,(\bar{x} \leq_K x \in X \Rightarrow \bar{x} = x) \tag{3.14}$$

and

$$(\{\bar{x}\} + K) \cap X = \{\bar{x}\}\,,\ i.e\,(x \leq_K \bar{x} \in X \Rightarrow \bar{x} = x) \tag{3.15}$$

respectively. Without loss of generality, for any nonempty partially ordered set X induced by convex cone, the minimal element can always be extended to the maximal element. So, we shall only discuss the minimal element of the set X henceforth.

Definition 3.14 *Let X be a nonempty subset of a partially ordered linear space with an ordering cone K . An element $\bar{x} \in X$ is called strongly minimal element of the set X if*

$$X \subseteq \{\bar{x}\} + K,\ i.e\,(\bar{x} \leq_K x\,\forall\, x \in X) \tag{3.16}$$

Definition 3.15 *Let Ω be a nonempty subset of a real linear space X and let*

$\bar{\lambda} > 0$. *Then for every* $x \in \Omega$, *the set*

$$cor(\Omega) = \left\{\bar{x} \in \Omega \mid \forall\, x \in X,\ \exists\, \bar{\lambda} > 0 : \forall\, \lambda \in [0, \bar{\lambda}],\ \bar{x} + \lambda x \in \Omega\right\}$$

is called the algebraic interior of Ω *or the core of* Ω.

Definition 3.16 *Let* X *be a nonempty partially ordered subset of* $\Re^n$ *with an ordering cone* K *which has a nonempty algebraic interior. An element* $\bar{x} \in X$ *is called weakly minimal element of the set* X *if*

$$(\{\bar{x} - cor(K)) \cap X = \emptyset\}\,. \tag{3.17}$$

We turn now to discuss MOP problems and the methods of solving them. But it would make sense if we get familiar with some nomenclatures involving MOP. In Applied Sciences, MOP is often referred to as Vector optimization problems. In Engineering, it is usually called Multicriteria OP while in Economics, it is often referred to as Multicriteria decision making Problems. The optimality concepts for these problems was first investigated by Pareto and Edgeworth[1]. In the fields of Economics and Engineering, minimal elements of a set are often referred to as the efficient solutions or Pareto optimal points or nondominated points.

Let X be a nonempty subset of $\Re^n$ and let $f : X \to \Re^m$ be a given vector function. Suppose that the image space $\Re^m$ is partially ordered in a natural way (i.e, $\Re^m_+$ is the ordering cone).

The general MOP in finite dimensional spaces can be written as

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \\ &\qquad\qquad x \in X \end{aligned} \tag{3.18}$$

Clearly, in the case $m = 1$, (3.18) reduces to a standard optimization problem with a single-valued function. We define the set of objective space (attainable outcomes) as

$$\{y \in \Re^m \mid y = f(x), x \in X\} = f(X) \tag{3.19}$$

The objective function and constraints are functions of a set of decision variables and parameters. When minimizing vector valued functions, we mean to find the minimal elements of the image set $f(X)$ with respect to the natural partial ordering. Practically, the minimal elements of the image set $f(X)$ do not play the central role but their pre-images. That is the set $X = \{x \mid x \in X\}$.

Definition 3.17 *$\bar{x} \in X$ is called a Pareto optimal point or a minimal solution of Problem* (3.18)*, if $f(\bar{x})$ is a minimal element of the image set $f(X)$ with respect to the natural partial ordering, i.e., there is no $x \in X$ with*

$$f_i(x) \leq f_i(\bar{x}), \qquad \forall\, i = 1, 2, ..., m$$

and

$$f(x) \neq f(\bar{x})$$

Definition 3.18 *$\bar{x} \in X$ is called a weakly Pareto optimal point or weakly minimal solution of Problem* (3.18) *if $f(\bar{x})$ is a weakly minimal element of the image set $f(X)$ with respect to the natural partial ordering. That is, there is no $x \in X$ with $f_i(x) < f_i(\bar{x})$ for all $x \in X$ and $i = 1, 2, ..., m$.*

Definition 3.19 *$\bar{x} \in X$ is called a strongly Pareto optimal point or strongly minimal solution of Problem* (3.18) *if $f(\bar{x})$ is a strongly minimal element of the image set $f(X)$ with respect to the natural partial ordering. That is, $f_i(\bar{x}) < f_i(x)$ for all $x \in X$ and $i = 1, 2, ..., m$.*

3.3 Methods of solving Multiobjective Optimization Problems

The purpose of Multiobjective optimization is to simultaneously optimize multiple conflicting objectives. Due to the conflicting nature of the objectives, a feasible solution optimizing all the objectives at the same time does not exist. Consequently, the goal of MOP is to obtain the nondominated set. For a Multiobjective optimization problems, a nondominated point in the objective space coincide to an efficient solution in decision space while the efficient solution is defined as a solution for which an improvement in one objective will continually worsening at least one of the other objectives. So, the set of all nondominated points forms the nondominated set in objective space.

The nondominated sets or objective function space cannot easily be computed in many cases. In some cases where it is possible to find all these points exactly, they might be of huge size (comprising many such points)[6]. Approximation methods for them are frequently used due to the complexity on the computations. Although, approximation does not produce an insignificant choice for the decision maker. In fact, there are many real life problems for which the decision maker may find it difficult to formulate the problem entirely and accurately. So, they tends to manage some basic solutions that are available. Having some approximated solutions would help in such situations and it would help to know if an exact method is really required and on the other hand, exploit such solution to improve the problem formulation (Ruzica and Wiecek [8]).

Approximation methods may have a specific goal. The full details on approximation classes can be found in [8]. They gave a comprehensive survey for the various cases of objectives, that is, when the numbers of objectives are two and when they are more. This survey covers more than 50 articles published since 1975. The book of Ehrgott [9]) makes a huge survey on the techniques of solving Multiobjective optimization integer programing. The discussion about different scalarization techniques were given in [10]. We shall only discuss the Weighted sum scalarization technique in this thesis work.

3.4 Weighted Sum Method

The sets of multiobjective can be scalarized into a single objective by pre-multiplying each objective with a user-supplied weight. This method is widely used because of its simplicity. However, the question about which weight to be used may not be well answered because the answer depends on the importance of each objective in the context of the problem and a scaling factor. Usually, the weight of an objective is chosen in proportion to the objectives relative importance in the problem. These weight coefficients can be normalized to 1, while this is not necessary in general. The weighted sum method to Problem (3.18) is reformulated as

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{m} w_i f_i(x) \\ &\text{subject to} \\ &\qquad\qquad x \in X \end{aligned} \tag{3.20}$$

where $w_i \geq 0$, $\forall\, i = 1, 2, ..., m$.

Apart from the difficulties in the computation, the scalarization method has the following shortcomings.

1. The relationship between the objective function weights and the objective function space is such that a uniform spread of weight parameters does not in general produce a uniform spread of points on the objective function space. What

we can be deduce from this fact is that all the points are grouped in certain parts of the Pareto front, while some (possibly significant) portions of the trade-off curve have not been produced.

Meanwhile, the following Theorem, gives us a huge motivation to use Weighted sum approach when the MOP under consideration is convex.

Theorem 3.5 *Let $w_1, w_2, ..., w_m$ be a given positive numbers, if $\bar{x}$ is the solution of the Scalarization problem in (3.20), then $\bar{x}$ is a properly Pareto optimal point or properly minimal point of the MOP (3.18).*

2. For a non-convex MOP, the non-convex parts of the Pareto set cannot be reached by minimizing convex combinations of the objective functions. By geometrical approach, an example can be made showing a weighted-sum method in two dimensions, i.e when there are exactly two objectives to minimize.

In the two-dimensional space the objective function is a line

$$y = w_1 f_1(x) + w_2 f_2(x) \tag{3.21}$$

from which it follows that $$f_2(x) = \frac{-w_1 f_1(x)}{w_2} + \tfrac{y}{w_2},$$

$$w_1 > 0, w_2 > 0.$$

The minimization of $wf(x)$ in the weighted sum method can be seen as an attempt to find the y value for which the line with slope $\frac{-w_1}{w_2}$ is tangent to the region S. It could be seen clearly that changing the weight parameters leads to possibly different touching points of the line to the feasible region. If the objective function space is convex, then one can calculate such points for different w vectors. Figure 3.1 is the geometric interpretation of the above discussion.

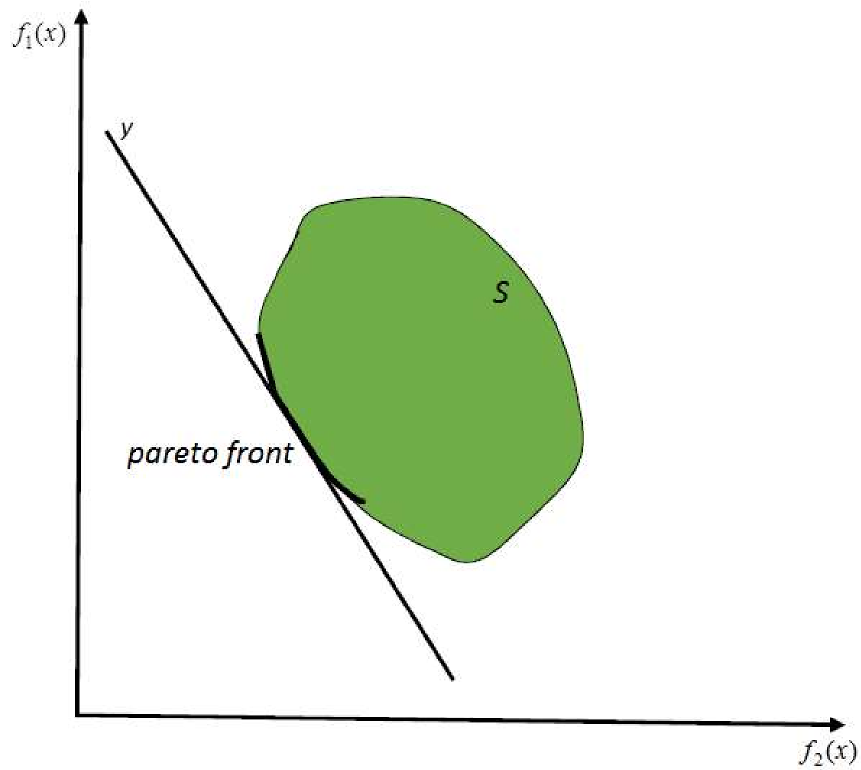

Figure 3.1: Illustration of the weighted sum method for when the objective function space is convex for two objectives.

Conversely, when the objective function space is non-convex, there are set of points that would be left out for any combinations of the weights (w_i). This is shown in Figure 3.2.

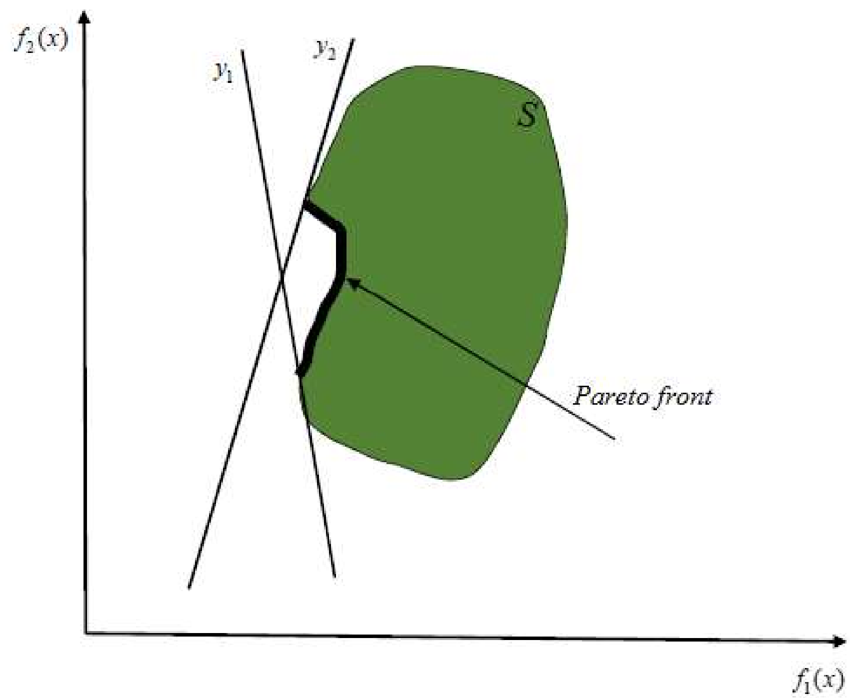

Figure 3.2: Illustration of the weighted sum method for when the objective function space is non-convex for two objectives.

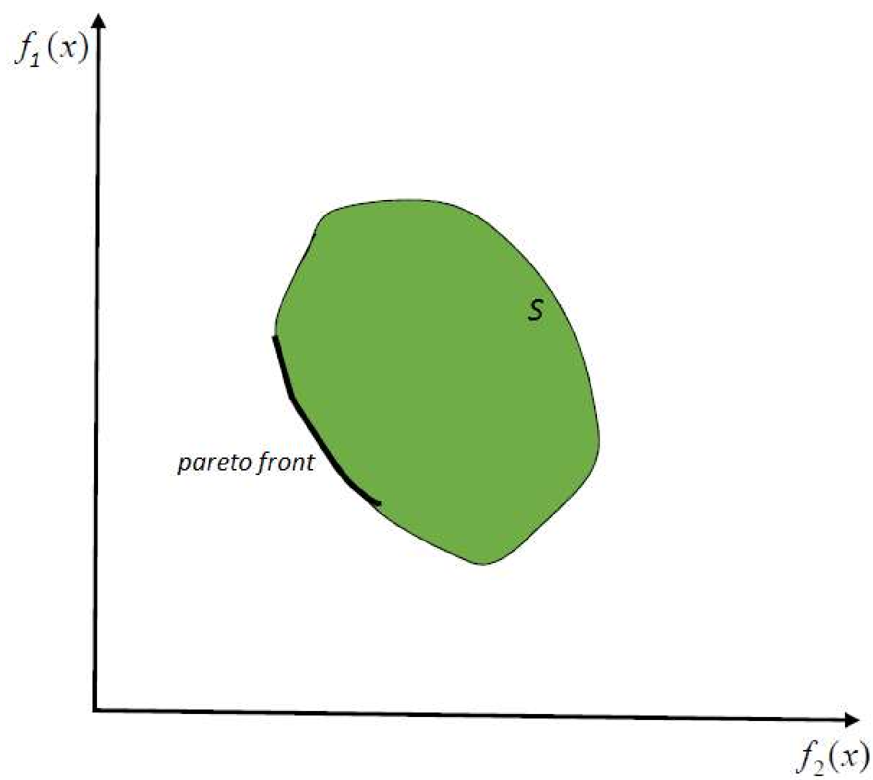

Figure 3.3: Example of objective function space for when the objectives are two

3.5 Conclusive Remark

In this Chapter, we made a brief explanation on convexity and quasiconvexity of a function. The concepts of optimization were discussed and in particular, MOP. We mentioned the existence theorems for minimal or maximal element of a set. The notions on optimality for MOP were equally discussed. We equally discussed Weighted sum scalarization approach for MOP. In the next Chapter, we shall give a brief literature review on MOP and control charts.

CHAPTER 4

SOME GENERALIZATIONS OF EWMA CONTROL PROCEDURE

Control chart is a statistical tool used in monitoring statistical processes. It is very powerful because of its ability to detect variations in a given process. The concept of control chart was first suggested by Walter A. Shewhart in the 1920s. Roberts[25] introduced EWMA Control Chart which is one of the advanced techniques used in monitoring statistical processes. Due to the high sensitivity of detecting small changes in a process mean or standard deviation, EWMA is widely used in the chemical industries where small process disturbances often result to serious financial penalties[21].

The performance evaluation of a control chart is measured by the average run length (ARL). The ARL of a charting structure is the average number of samples to be taken before a false alarm is detected in the process. The performance can be evaluated by two values ARL_0 and ARL_1. ARL_0 is the expected number of samples before an out-of-control false alarm is detected when the process is at in-control state while ARL_1 is the expected number of samples before an out-of-control false alarm occurs when the process is shifted to an out-of-control state. A chart is considered to be more effective than the other chart if it has a smaller ARL_1 (out-of-control ARL) values at more points Wu et al.[39].

4.1 The EWMA control chart for independent and identically distributed observations (i.i.d)

The exponentially moving average control chart for i.i.d is defined as

$$Z_t = rX_t + (1-r)Z_{t-1} \tag{4.1}$$

where $X_t \sim \mathcal{N}(\mu, \sigma^2)$, t is the sample number and r is the constant such that $0 < r \leq 1$.

(4.1) can be written as

$$Z_t = \begin{cases} Z_0 & when\, t = 0 \\ r\sum_{i=0}^{t-1}(1-r)^i X_{t-i} + (1-r)^t Z_0 & for\, t = 1, 2, ... \end{cases} \tag{4.2}$$

By the sum of a geometric progression, the term

$$r\sum_{i=0}^{t-1}(1-r)^i + (1-r)^t = 1$$

The choice of r determines the decline of the weights. For all the possible choices of r, more recent observations always have more weight in the computation of Z_t than the older observations. if $r = 1$, then $Z_t = X_t$ and the EWMA control chart will then behave as the Shewhart control chart. If $r \to 0$, then the most recent observation have a small weight. Now, if we take $Z_0 = \mu$, then from (4.2), $\mathbf{E}(Z_t) = \mu$ and the variance $V(Z_t)$ is given by

$$\sigma^2_{Z_t} = \sigma^2_X \left(\frac{r}{2-r}\right)\left[1-(1-r)^{2t}\right] \tag{4.3}$$

Using (4.3), The control structure of the EWMA chart which includes the upper control limit (UCL), center line(CL), and lower control limit (LCL) is defined as

$$LCL = \mu_0 - L\sigma\sqrt{\left(\frac{r}{2-r}\right)(1-(1-r)^{2t})} \tag{4.4}$$

$$CL = \mu_0$$

$$UCL = \mu_0 + L\sigma\sqrt{\left(\frac{r}{2-r}\right)(1-(1-r)^{2t})} \tag{4.5}$$

On the basis of 10,000 Monte Carlo simulation from standard normal distribution, it is assumed that the in-control process is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, that is $X_t \sim N(0,1)$. So, after a certain time t, a random shift δ occurs in the process mean. This shifts process mean from μ to a new mean level μ_1. That is, $\mu_1 = \mu + \delta\sigma$ or $\delta = \frac{|\mu_1 - \mu|}{\sigma_0.}$ Where δ represent the amount of shift n the process mean level. The ARL values for EWMA have been calculated and reported in Table (4.1).

Table 4.1: ARL values for classic EWMA control charts for when the $ARL_0 = 500$.

$r \rightarrow$	0.1	0.25	0.5	0.75
$\delta \downarrow$ $L \rightarrow$	2.8248	3.0000	3.0690	3.0829
0	500.2374	500.8845	500.3367	500.7104
0.25	103.4474	168.7243	252.0264	319.4916
0.5	28.5189	47.1542	88.5273	137.1711
0.75	13.4946	19.0963	34.9501	61.0826
1	8.283	10.4003	17.0879	30.1124
1.25	5.6177	6.6975	9.6523	16.2177
1.5	4.144	4.7335	6.1889	9.6582
1.75	3.2685	3.6501	4.4372	6.3557
2	2.6749	2.9513	3.3964	4.4749
2.25	2.2497	2.4597	2.7299	3.3092
2.5	1.9083	2.0715	2.2397	2.5891
2.75	1.6861	1.8134	1.9236	2.1326
3	1.5175	1.6207	1.6974	1.8038
3.25	1.3779	1.4602	1.513	1.5813
3.5	1.2613	1.3305	1.3706	1.4073
3.75	1.1794	1.2384	1.2623	1.2836
4	1.1135	1.1532	1.1704	1.1821

Let $X_1, X_2, ...$ be an independent random variables observed over time t. Assume that the X_t for each t is normally distributed with mean μ and variance σ^2 i.e. $X_t \sim \mathcal{N}(\mu, \sigma^2)$. Let $\bar{X}_t$ be the sample mean of the independent random variable with n observations taking over time t. Then, $\bar{X}_t$ is normally distributed i.e. $X_t \sim \mathcal{N}(\mu, \sigma^2/n)$. So, all chart will be constructed for a standard normal variable

$$Y_t = \frac{\bar{X}_t - \mu}{\sigma^2/n} \tag{4.6}$$

where $Y_t \sim \mathcal{N}(0,1)$.

Following (4.1), let $Z_t = [A^t, D^t] \begin{bmatrix} \beta Z_{t-1} \\ \gamma Y_t \end{bmatrix}$

$= A^t \beta Z_{t-1} + \gamma D^t Y_t$

where A and D are 2x1 matrices given by $A = \begin{bmatrix} \zeta \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\beta = (1-r)$ and $\gamma = r$ for some $r \in (0,1]$.

Thus, we get

$$Z_t = \min \beta \{\zeta Z_{t-1}, Z_{t-1}\} + \gamma Y_t \tag{4.7}$$

where $\zeta \in [0,1]$. With $\zeta = 1$, (4.7) reduces to classical EWMA in (4.1).

On the basis of 10,000 Monte Carlo simulation from standard normal distribution, it is assumed that the in-control process is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, that is $Y_t \sim N(0,1)$. So, after a certain time t, a random shift δ occurs in the process mean. This shifts process mean from μ to a new mean level μ_1. That is, $\mu_1 = \mu + \delta\sigma$ or $\delta = \frac{|\mu_1 - \mu|}{\sigma_0.}$ Where δ represent the amount of shift n the process mean level. The ARL values for min-EWMA for different values of ζ at $r = 0.1, r = 0.25, r = 0.5$ and $r = 0.75$ and have been calculated and reported in the Tables below

Table 4.2: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.1$, $\zeta = 0$ and $L = 2.7499$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	499.0445	504.4649	23	141	341	689	1512.1
0.25	36.8102	43.1996	2	6	22	52	124
0.5	2.1299	0.6105	2	2	2	2	3
0.75	1.9748	0.1580	2	2	2	2	2
1	1.9604	0.1950	2	2	2	2	2
1.25	1.9296	0.2558	1	2	2	2	2
1.5	1.8911	0.3115	1	2	2	2	2
1.75	1.8437	0.3632	1	2	2	2	2
2	1.7666	0.4230	1	2	2	2	2
2.25	1.6909	0.4621	1	1	2	2	2
2.5	1.5906	0.4917	1	1	2	2	2
2.75	1.4974	0.5000	1	1	1	2	2
3	1.4064	0.4912	1	1	1	2	2
3.25	1.3122	0.4634	1	1	1	2	2
3.5	1.2218	0.4155	1	1	1	1	2
3.75	1.1573	0.3641	1	1	1	1	2
4	1.0981	0.2975	1	1	1	1	2

Table 4.3: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.1$, $\zeta = 0.3$ and $L = 2.7199$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	501.4481	494.858	24	145	352	700	1499
0.25	42.763	44.69939	2	10	28	60	135
0.5	3.0307	2.257314	2	2	2	3	6
0.75	2.0061	0.2299306	2	2	2	2	2
1	1.9579	0.2013247	2	2	2	2	2
1.25	1.9301	0.2549912	1	2	2	2	2
1.5	1.8839	0.3203608	1	2	2	2	2
1.75	1.8345	0.3716498	1	2	2	2	2
2	1.7629	0.4253254	1	2	2	2	2
2.25	1.6804	0.4663449	1	1	2	2	2
2.5	1.5895	0.4919491	1	1	2	2	2
2.75	1.4832	0.4997427	1	1	1	2	2
3	1.3941	0.488681	1	1	1	2	2
3.25	1.2995	0.4580619	1	1	1	2	2
3.5	1.2212	0.4150756	1	1	1	1	2
3.75	1.1535	0.3604868	1	1	1	1	2
4	1.1046	0.3060526	1	1	1	1	2

Table 4.4: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.1$, $\zeta = 0.6$ and $L = 2.6845$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.77	497.4976	23	143	351	699	1499.1
0.25	48.9863	47.71911	4	14	34	68	145
0.5	6.0298	5.518655	2	3	4	7	17
0.75	2.5327	0.8128941	2	2	2	3	4
1	2.0482	0.3761528	2	2	2	2	3
1.25	1.9307	0.2747818	1	2	2	2	2
1.5	1.8762	0.3296732	1	2	2	2	2
1.75	1.8243	0.3805838	1	2	2	2	2
2	1.7511	0.4323974	1	2	2	2	2
2.25	1.6664	0.4715223	1	1	2	2	2
2.5	1.5744	0.4944584	1	1	2	2	2
2.75	1.4713	0.4992006	1	1	1	2	2
3	1.3816	0.4858035	1	1	1	2	2
3.25	1.2872	0.4524783	1	1	1	2	2
3.5	1.2113	0.4082511	1	1	1	1	2
3.75	1.1467	0.3538243	1	1	1	1	2
4	1.0978	0.2970589	1	1	1	1	2

Table 4.5: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.1$, $\zeta = 0.94$ and $L = 2.6663$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.1179	513.6991	18	134	335	699	1531.05
0.25	69.2938	65.80626	6	23	50	93	202
0.5	17.9903	14.78278	3	8	14	24	46
0.75	7.7623	5.138564	2	4	7	10	18
1	4.7137	2.615878	2	3	4	6	10
1.25	3.3726	1.646098	1	2	3	4	6
1.5	2.6551	1.148336	1	2	2	3	5
1.75	2.2313	0.8645664	1	2	2	3	4
2	1.9463	0.7194915	1	1	2	2	3
2.25	1.7574	0.6224018	1	1	2	2	3
2.5	1.5891	0.5546999	1	1	2	2	2
2.75	1.475	0.52173	1	1	1	2	2
3	1.3776	0.4921813	1	1	1	2	2
3.25	1.2855	0.4530014	1	1	1	2	2
3.5	1.1969	0.3984297	1	1	1	1	2
3.75	1.1395	0.3467734	1	1	1	1	2
4	1.0841	0.2775516	1	1	1	1	2

Table 4.6: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.25$, $\zeta = 0$ and $L = 2.8999$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	498.5016	502.6896	27	142	343	685	1523.05
0.25	88.8291	88.01237	6	26	62	123	263
0.5	22.4834	21.45697	2	7	16	31	66
0.75	6.7724	6.347395	2	2	4	9	20
1	2.756	1.556505	2	2	2	3	6
1.25	2.0491	0.4408045	1	2	2	2	3
1.5	1.9246	0.2924438	1	2	2	2	2
1.75	1.8776	0.3292857	1	2	2	2	2
2	1.8123	0.3904919	1	2	2	2	2
2.25	1.7409	0.4381625	1	1	2	2	2
2.5	1.6491	0.4772756	1	1	2	2	2
2.75	1.5601	0.4963997	1	1	2	2	2
3	1.4652	0.4988124	1	1	1	2	2
3.25	1.365	0.4814542	1	1	1	2	2
3.5	1.2699	0.4439298	1	1	1	2	2
3.75	1.1962	0.3971415	1	1	1	1	2
4	1.1248	0.3305085	1	1	1	1	2

Table 4.7: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.25$, $\zeta = 0.3$ and $L = 2.877$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	502.4062	517.2124	25	140	344	687	1542.1
0.25	91.8974	89.51753	7	28	65	125	273
0.5	25.0972	23.67018	3	8	18	34	72
0.75	8.5265	7.560213	2	3	6	11	23
1	3.5417	2.224805	2	2	3	4	8
1.25	2.3441	0.8405747	1	2	2	3	4
1.5	2.0025	0.438535	1	2	2	2	3
1.75	1.8842	0.361391	1	2	2	2	2
2	1.8152	0.3914898	1	2	2	2	2
2.25	1.7304	0.4437742	1	1	2	2	2
2.5	1.6469	0.4779574	1	1	2	2	2
2.75	1.5464	0.4978673	1	1	2	2	2
3	1.4502	0.4975387	1	1	1	2	2
3.25	1.3583	0.4795249	1	1	1	2	2
3.5	1.2684	0.443149	1	1	1	2	2
3.75	1.1905	0.3927151	1	1	1	1	2
4	1.1353	0.3420608	1	1	1	1	2

Table 4.8: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.25$, $\zeta = 0.6$ and $L = 2.849$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.7473	516.7254	23.95	139	342	685	1540.05
0.25	96.4675	93.88315	7	30	68	132	282.05
0.5	27.6411	25.56565	3	10	20	38	78
0.75	10.5085	8.877894	2	4	8	14	28
1	4.8998	3.269194	2	3	4	6	11
1.25	3.0741	1.489843	1	2	3	4	6
1.5	2.383	0.8655553	1	2	2	3	4
1.75	2.0446	0.5951859	1	2	2	2	3
2	1.8672	0.4890686	1	2	2	2	3
2.25	1.7334	0.4713244	1	1	2	2	2
2.5	1.6391	0.4852574	1	1	2	2	2
2.75	1.5333	0.4993156	1	1	2	2	2
3	1.4394	0.4963389	1	1	1	2	2
3.25	1.3469	0.4760074	1	1	1	2	2
3.5	1.2599	0.4386014	1	1	1	2	2
3.75	1.1837	0.3872587	1	1	1	1	2
4	1.1281	0.3342178	1	1	1	1	2

Table 4.9: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.25$, $\zeta = 0.94$ and $L = 2.9202$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.5094	508.1203	25	143	344	695	1505.05
0.25	134.3155	131.1089	8	40	94	186	396
0.5	38.7138	35.68493	4	13	27	53	110
0.75	16.023	13.12013	3	7	12	21	42
1	8.6859	6.22369	2	4	7	11	21
1.25	5.5295	3.499022	2	3	5	7	12
1.5	3.9371	2.202142	1	2	4	5	8
1.75	3.0793	1.537676	1	2	3	4	6
2	2.5404	1.204123	1	2	2	3	5
2.25	2.1534	0.9600837	1	1	2	3	4
2.5	1.8666	0.7862991	1	1	2	2	3
2.75	1.6747	0.6686738	1	1	2	2	3
3	1.5231	0.5955685	1	1	1	2	2
3.25	1.3921	0.5300808	1	1	1	2	2
3.5	1.2846	0.468428	1	1	1	2	2
3.75	1.2053	0.4093515	1	1	1	1	2
4	1.1304	0.3388323	1	1	1	1	2

Table 4.10: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.5$, $\zeta = 0$ and $L = 2.9688$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.0228	500.4182	27	141	343	687	1514
0.25	150.7961	147.0402	8	44	104	211	451
0.5	51.2322	50.6813	4	16	35.5	71	152
0.75	21.0323	19.49229	2	7	15	29	60
1	10.1869	8.756383	2	4	8	14	28
1.25	5.7179	4.413147	2	3	4	7	14
1.5	3.5998	2.348785	1	2	3	4	8
1.75	2.6187	1.320562	1	2	2	3	5
2	2.1331	0.8384836	1	2	2	2	4
2.25	1.871	0.6013278	1	2	2	2	3
2.5	1.7052	0.5214596	1	1	2	2	2
2.75	1.5971	0.5106835	1	1	2	2	2
3	1.4943	0.5045721	1	1	1	2	2
3.25	1.3901	0.4886163	1	1	1	2	2
3.5	1.2936	0.4556532	1	1	1	2	2
3.75	1.2187	0.4136271	1	1	1	1	2
4	1.1408	0.3478325	1	1	1	1	2

Table 4.11: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.5$, $\zeta = 0.3$ and $L = 2.963$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	501.9257	503.7051	28	145	348	687	1497.1
0.25	161.4867	161.3631	9	46	112	221	485
0.5	55.0598	53.65122	4	17	39	75	161
0.75	22.9156	21.74844	3	8	16	31	66
1	11.2132	9.844025	2	4	8	15	31
1.25	6.3484	4.919658	2	3	5	8	16
1.5	4.0284	2.700912	1	2	3	5	9
1.75	2.912	1.605837	1	2	2	4	6
2	2.3013	1.034517	1	2	2	3	4
2.25	1.9546	0.7463207	1	1	2	2	3
2.5	1.7583	0.6034215	1	1	2	2	3
2.75	1.621	0.548807	1	1	2	2	2
3	1.4972	0.5169322	1	1	1	2	2
3.25	1.3939	0.4935252	1	1	1	2	2
3.5	1.2975	0.4584918	1	1	1	2	2
3.75	1.223	0.4165193	1	1	1	1	2
4	1.1498	0.3568931	1	1	1	1	2

Table 4.12: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.5$, $\zeta = 0.6$ and $L = 2.970$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	501.2805	501.1642	28	145	348	689.25	1508
0.25	177.9739	178.1295	10	51	124	244	532
0.5	61.2208	60.2728	5	19	43	84	180
0.75	25.5591	24.29427	3	9	18	35	75
1	12.6612	11.30061	2	5	9	17	35
1.25	7.2141	5.661366	2	3	5	10	18
1.5	4.6377	3.231297	1	2	4	6	11
1.75	3.3248	1.966848	1	2	3	4	7
2	2.5868	1.335906	1	2	2	3	5
2.25	2.1303	0.970369	1	2	2	3	4
2.5	1.8559	0.7420177	1	1	2	2	3
2.75	1.6779	0.6390559	1	1	2	2	3
3	1.5237	0.565217	1	1	1	2	2
3.25	1.4061	0.5161487	1	1	1	2	2
3.5	1.302	0.4662808	1	1	1	2	2
3.75	1.2255	0.4203182	1	1	1	1	2
4	1.1517	0.3587478	1	1	1	1	2

Table 4.13: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.5$, $\zeta = 0.94$ and $L = 3.0419$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.8315	497.0191	25	145.75	345	698	1486.1
0.25	247.2262	243.5296	13	71	173	347	726.05
0.5	103.817	104.1375	7	31	72	142	310
0.75	46.831	46.29398	3	14	33	64	137
1	23.3988	22.19741	2	8	17	32	66
1.25	12.8513	11.61575	2	5	9	18	36
1.5	7.8122	6.82403	1	3	6	10	21
1.75	5.1247	4.076372	1	2	4	7	13
2	3.6853	2.66499	1	2	3	5	9
2.25	2.7782	1.748288	1	2	2	3	6
2.5	2.2229	1.24546	1	1	2	3	5
2.75	1.8821	0.928378	1	1	2	2	4
3	1.6476	0.734485	1	1	2	2	3
3.25	1.474	0.6121777	1	1	1	2	2
3.5	1.3413	0.517147	1	1	1	2	2
3.75	1.2463	0.450618	1	1	1	1	2
4	1.1605	0.3735689	1	1	1	1	2

Table 4.14: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.75$, $\zeta = 0$ and $L = 3.0468$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	500.8315	497.0191	25	145.75	345	698	1486.1
0.25	247.2262	243.5296	13	71	173	347	726.05
0.5	103.817	104.1375	7	31	72	142	310
0.75	46.831	46.29398	3	14	33	64	137
1	23.3988	22.19741	2	8	17	32	66
1.25	12.8513	11.61575	2	5	9	18	36
1.5	7.8122	6.82403	1	3	6	10	21
1.75	5.1247	4.076372	1	2	4	7	13
2	3.6853	2.66499	1	2	3	5	9
2.25	2.7782	1.748288	1	2	2	3	6
2.5	2.2229	1.24546	1	1	2	3	5
2.75	1.8821	0.928378	1	1	2	2	4
3	1.6476	0.734485	1	1	2	2	3
3.25	1.474	0.6121777	1	1	1	2	2
3.5	1.3413	0.517147	1	1	1	2	2
3.75	1.2463	0.450618	1	1	1	1	2
4	1.1605	0.3735689	1	1	1	1	2

Table 4.15: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.75$, $\zeta = 0.3$ and $L = 3.052$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	502.6061	495.976	27	148	357	696	1479
0.25	266.9111	268.332	15	76	183	373	785
0.5	111.0938	109.9027	7	32	77	155	329
0.75	50.3898	49.10228	4	15	35	69	148.05
1	24.9585	24.00151	2	8	17	34	74
1.25	13.8487	12.70973	2	5	10	19	40
1.5	8.2696	7.268645	1	3	6	11	23
1.75	5.3607	4.287964	1	2	4	7	14
2	3.8443	2.808068	1	2	3	5	9
2.25	2.8634	1.899395	1	2	2	4	7
2.5	2.3051	1.326043	1	1	2	3	5
2.75	1.9547	1.026817	1	1	2	2	4
3	1.6827	0.8046648	1	1	2	2	3
3.25	1.5004	0.6467161	1	1	1	2	3
3.5	1.3538	0.5394948	1	1	1	2	2
3.75	1.2637	0.4703025	1	1	1	2	2
4	1.1784	0.3957134	1	1	1	1	2

Table 4.16: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.75$, $\zeta = 0.6$ and $L = 3.060$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	501.1368	496.8814	26	148	354	694	1472
0.25	284.396	284.004	16	82	198	397	828.1
0.5	120.4098	119.0246	8	35	84	167	359
0.75	54.585	53.45713	4	17	38	75	161
1	26.841	25.8441	2	8	19	37	81
1.25	14.6716	13.46193	2	5	11	20	42
1.5	8.7935	7.774117	1	3	6	12	24
1.75	5.6775	4.57458	1	2	4	7	15
2	4.0682	3.040769	1	2	3	5	10
2.25	3.0159	2.071926	1	2	2	4	7
2.5	2.4087	1.450198	1	1	2	3	5
2.75	2.0253	1.121421	1	1	2	2	4
3	1.7234	0.8658911	1	1	2	2	3
3.25	1.5274	0.6923129	1	1	1	2	3
3.5	1.3706	0.5754031	1	1	1	2	2
3.75	1.2734	0.4926223	1	1	1	2	2
4	1.1827	0.4051384	1	1	1	1	2

Table 4.17: The ARL, SDRL values and run length quantiles for min-EWMA control charts at $r = 0.75$, $\zeta = 0.94$ and $L = 3.0799$ when $ARL_0 = 500$.

δ	ARL	$SDRL$	$P_{0.05}$	$P_{0.25}$	$P_{0.5}$	$P_{0.75}$	$P_{0.95}$
0	499.4993	496.1207	26	146	354.5	694	1465.1
0.25	312.8955	310.0553	17	92	217	435	919
0.5	134.0805	132.4019	8	39	95	185	402.05
0.75	61.2573	60.95485	4	18	42	84	180
1	29.7794	28.5461	2	9	21	41	89
1.25	16.2067	15.02944	2	5	12	22	46
1.5	9.5936	8.541331	1	4	7	13	27
1.75	6.1732	5.126152	1	3	5	8	16
2	4.3762	3.310554	1	2	3	6	11
2.25	3.219	2.27239	1	2	3	4	8
2.5	2.5648	1.642702	1	1	2	3	6
2.75	2.1387	1.270521	1	1	2	3	5
3	1.7922	0.959537	1	1	2	2	4
3.25	1.5747	0.7634646	1	1	1	2	3
3.5	1.3985	0.6293944	1	1	1	2	3
3.75	1.2907	0.526518	1	1	1	2	2
4	1.194	0.4289317	1	1	1	1	2

4.1.1 Results and Discussion

In this section, we discussed classical EWMA control chart. The performance measure in terms of ARL was investigated and reported in table (4.1). Also, we propose min-EWMA control scheme. It is the control scheme that mimics the classical EWMA control chart. Different values of scaling parameter ζ have been used to evaluate the performance measure (ARL values) of this scheme. The ARL tables for min-EWMA control scheme for different values of ζ were given in table (4.2), (4.3),(4.4),(4.5),(4.6),(4.7),(4.8),(4.9), (4.10),(4.11),(4.9),(4.9),(4.12),(4.13),(4.14),(4.15), (4.16) and (4.17). The ARL curves given in (4.1) indicated that the small values of ζ give a better ARL values. By the definition of min-EWMA control scheme, it reduces to classical EWMA control chart when $\zeta = 1$. Of course, it has exactly ARL values as classical EWMA control chart at $\zeta = 1$. The motivation for the min-EWMA control scheme is that, the case where a very small shift in a process is to be located at the earlier stage, min-EWMA is a better option.

Below, we present the graphs showing the effects of ζ for various values of weighing parameter r. For clarity, we consider the first four mean shifts (δ values) and plot it against the ARL values.

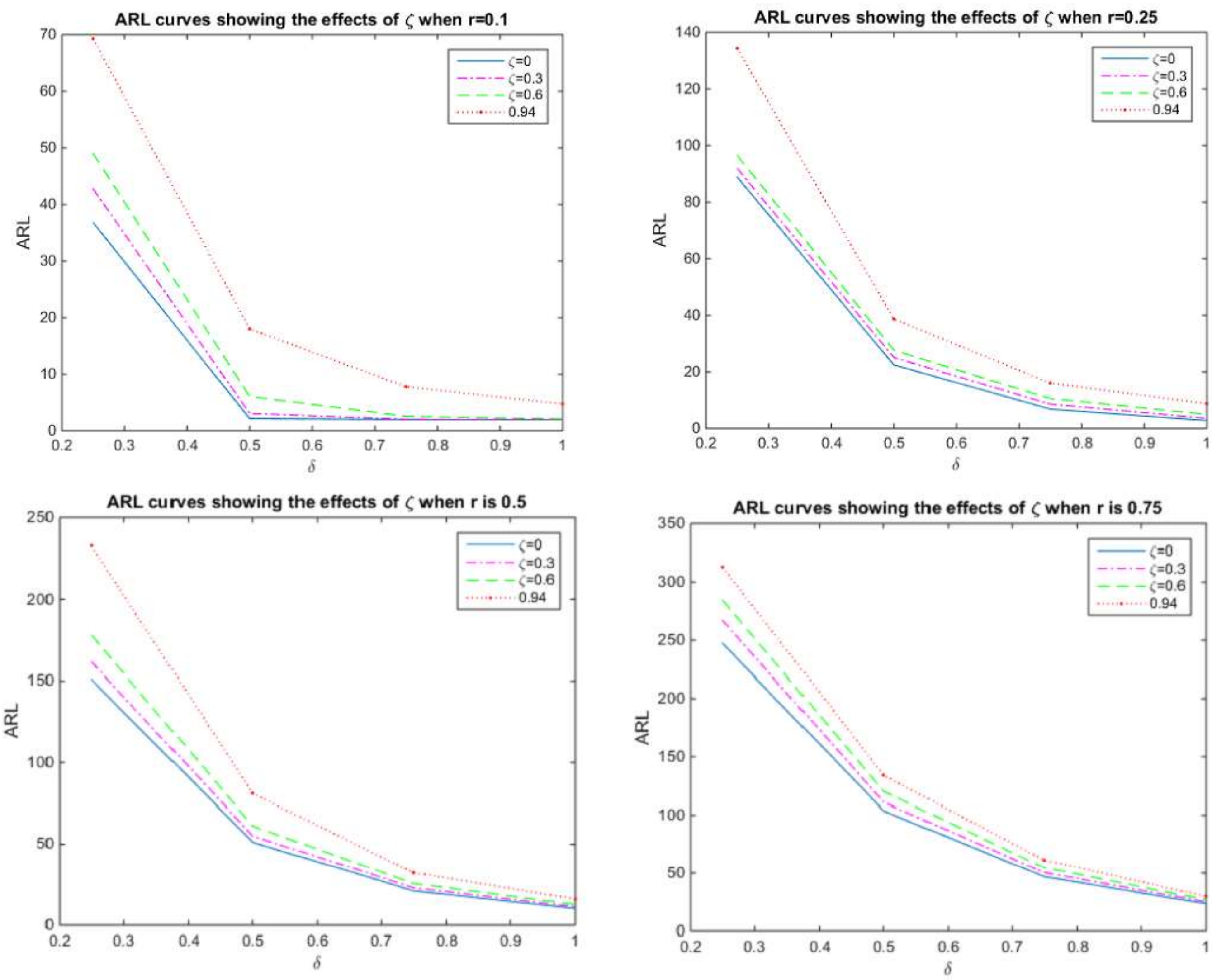

Figure 4.1: Performance comparison of control charts under min-EWMA control scheme

4.2 The Fast Initial Response Features for control charts (FIR)

This procedure was introduced by Lucas and Crosier[32] to improve the sensitivity of a Cumulative Sum (CUSUM) control chart at the process start up. Increased sensitivity at process start up would be desirable if the corrective action did not reset the mean to the target value. FIR or headstart sets the starting values C_0^+ and C_0^- equal to some nonzero value, e.g setting $C_0^+ = H/2 = C_0^-$, where H is the control limit for a CUSUM procedure. This is called a 50% headstart.

Steiner[40] proposed the time varying FIR$_{adj}$ features for EWMA procedure. It is given below.

Let

$$\text{FIR}_{adj} = 1 - (1 - f)^{1+a(t-1)} \tag{4.8}$$

With the setup in (4.8), the FIR$_{adj}$ makes the control limits for the first sample point (t=1) a proportion f, of the original distance from the starting value. The effect of the FIR$_{adj}$ reduces with time to ensure that the asymptotic properties of EWMA chart is not violated. Steiner sets $a = (-2/log(1 - f) - 1)/19$ so that the FIR$_{adj}$ has very little effect after 20th observation. i.e, FIR$_{adj}$ at 20th observation is ≈ 0.99. By setting $f = 0.5$ gives that $a = 0.3$. This will pave way for the quick detection of problems in the process startup. So, the respective upper and lower control limits for EWMA procedure under this scheme becomes

$$\mu_0 \pm L\sigma(1-(1-f)^{1+a(t-1)})\sqrt{\frac{r}{2-r}(1-(1-r)^{2t})} \tag{4.9}$$

An ARL evaluation shows that with $t = 1, f = 0.5$ and $a = 0.3$, when (4.8) is used in CUSUM procedure, it is equivalent to 50% headstart.

On the basis of 10,000 Monte Carlo simulation from standard normal distribution, it is assumed that the in-control process is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, that is $X_t \sim N(0,1)$. So, after a certain time t, a random shift δ occurs in the process mean. This shifts process mean from μ to a new mean level μ_1. That is, $\mu_1 = \mu + \delta\sigma$ or $\delta = \frac{|\mu_1 - \mu|}{\sigma_0.}$ Where δ represent the amount of shift n the process mean level. The ARL values for FIR-EWMA have been calculated and reported in Table (4.18).

Table 4.18: ARL values for Fast Initial Response EWMA control charts for when the $ARL_0 = 500$.

$r \rightarrow$ $\delta \downarrow \; L \rightarrow$	0.1 2.9100	0.25 3.0758	0.5 3.1430	0.75 3.1600
0	500.6636	500.3663	500.3632	500.1802
0.25	88.83	152.6688	230.0517	293.4516
0.5	21.4226	34.7766	66.2246	109.9724
0.75	8.867	11.786	20.8215	36.5382
1	4.7244	5.3801	7.5972	13.648
1.25	3.0346	3.2554	3.8366	5.7751
1.5	2.1911	2.3156	2.4892	3.0283
1.75	1.7164	1.7982	1.85	2.0086
2	1.4661	1.5155	1.55	1.5835
2.25	1.2778	1.316	1.3336	1.383
2.5	1.1783	1.2018	1.2142	1.2251
2.75	1.1078	1.1258	1.1382	1.142
3	1.0668	1.0811	1.0868	1.0847
3.25	1.0393	1.045	1.048	1.0543
3.5	1.0187	1.0221	1.0244	1.0266
3.75	1.0103	1.0134	1.0141	1.0159
4	1.0044	1.0063	1.007	1.0084

Haq et.al[41] modified the FIR_{adj} given in (4.8) by adding the term $1 + 1/t$ as power to it.

He obtained

$$\text{MFIR}_{adj} = \{1 - (1 - f)^{1+a(t-1)}\}^{1+1/t} \tag{4.10}$$

The respective upper and lower control limits for EWMA procedure under this scheme becomes

$$\mu_0 \pm L\sigma\{1 - (1 - f)^{1+a(t-1)}\}^{1+1/t}\sqrt{\frac{r}{2-r}(1 - (1 - r)^{2t})} \tag{4.11}$$

On the basis of 10,000 Monte Carlo simulation from standard normal distribution, it is assumed that the in-control process is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, that is $X_t \sim N(0, 1)$. So, after a certain time t, a random shift δ occurs in the process mean. This shifts process mean from μ to a new mean level μ_1. That is, $\mu_1 = \mu + \delta\sigma$ or $\delta = \frac{|\mu_1 - \mu|}{\sigma_0.}$ Where δ represent the amount of shift n the process mean level. The ARL values for MFIR-EWMA have been calculated and reported in table (4.19). An ARL evaluation shows that with $t = 1, f = 0.5$ and $a = 0.3$, when (4.10) is used in CUSUM procedure, it is equivalent to 25% headstart.

Table 4.19: ARL values for Modified Fast Initial Response EWMA control charts for when the $ARL_0 = 500$.

$r \rightarrow$ $\delta \downarrow \;\; L \rightarrow$	0.1 2.947	0.25 3.102	0.5 3.1759	0.75 3.1959
0	500.3397	500.0862	500.5418	500.0053
0.25	74.7434	137.654	218.9628	286.7315
0.5	15.8143	28.9738	55.4756	94.1858
0.75	6.0543	8.0035	15.2761	28.7075
1	3.2248	4.0031	5.1106	8.8399
1.25	2.10328	2.1847	2.4729	3.3494
1.5	1.56616	1.6012	1.6292	1.8208
1.75	1.30682	1.3211	1.3242	1.3751
2	1.16632	1.1777	1.1788	1.1867
2.25	1.09398	1.0998	1.0991	1.0993
2.5	1.05218	1.0548	1.0551	1.0548
2.75	1.0263	1.0289	1.0295	1.031
3	1.01304	1.0147	1.0152	1.0156
3.25	1.00678	1.0135	1.0123	1.0234
3.5	1.00304	1.0036	1.0035	1.0036
3.75	1.0012	1.0032	1.0003	1.0024
4	1.0005	1.0007	1.0007	1.0007

We provide the general time varying FIR feature ($GFIR_{adj}$) for control charts such that all percentiles of headstarts could easily be obtained. We consider the exponentially decreasing function

$$g(t) = 1 - (1 - f)^{(\phi+\psi)t-\psi} \tag{4.12}$$

where $0 < f < 1, \psi \in \Re^{+}$ and $0 < \phi < \infty$. Following Steiner[40], we let $\psi = (-2/log(1-f) - 1)/19$ (log with base 10). At $\phi = 1/t$, this is equivalent to FIR$_{adj}$ and at $\phi = \psi(t-1) - t$, it is equivalent to MFIR$_{adj}$. The respective upper and lower control limits for EWMA procedure under this scheme is

$$\mu_0 \pm L\sigma(1 - (1 - f)^{(\phi+\psi)t-\psi})\sqrt{\frac{r}{2-r}(1 - (1 - r)^{2t})} \tag{4.13}$$

On the basis of 10,000 Monte Carlo simulation from standard normal distribution, it is assumed that the in-control process is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, that is $X_t \sim N(0, 1)$. So, after a certain time t, a random shift δ occurs in the process mean. This shifts process mean from μ to a new mean level μ_1. That is, $\mu_1 = \mu + \delta\sigma$ or $\delta = \frac{|\mu_1 - \mu|}{\sigma_0.}$ Where δ represent the amount of shift n the process mean level. The ARL values for GFIR-EWMA have been calculated and reported in Table (4.20).

Table 4.20: ARL values for Generalized Fast Initial Response EWMA control charts at $\phi = 0.05$ when $ARL_0 = 500$.

$r \rightarrow$ $\delta \downarrow$ $L \rightarrow$	0.1 3.094	0.25 3.255	0.5 3.3299	0.75 3.357
0	500.7811	500.4117	500.0677	500.3902
0.25	70.5289	129.7159	206.0643	265.6224
0.5	14.5205	24.5184	45.7626	80.2204
0.75	5.3603	6.6713	10.881	21.0246
1	3.0222	3.1848	3.9386	6.0709
1.25	2.1402	2.1977	2.2954	2.8237
1.5	1.7059	1.7457	1.7673	1.8825
1.75	1.4898	1.5215	1.5383	1.5732
2	1.3509	1.3843	1.4003	1.4147
2.25	1.2558	1.2811	1.293	1.3013
2.5	1.1653	1.1873	1.1996	1.2065
2.75	1.1172	1.1325	1.1411	1.1451
3	1.0753	1.0874	1.0931	1.0958
3.25	1.044	1.0519	1.0568	1.058
3.5	1.0244	1.0294	1.0318	1.0327
3.75	1.0134	1.0165	1.0181	1.0185
4	1.0078	1.0091	1.0105	1.0108

To visualize the performance comparison of these control schemes, we present the graphical representation in figure (4.2).

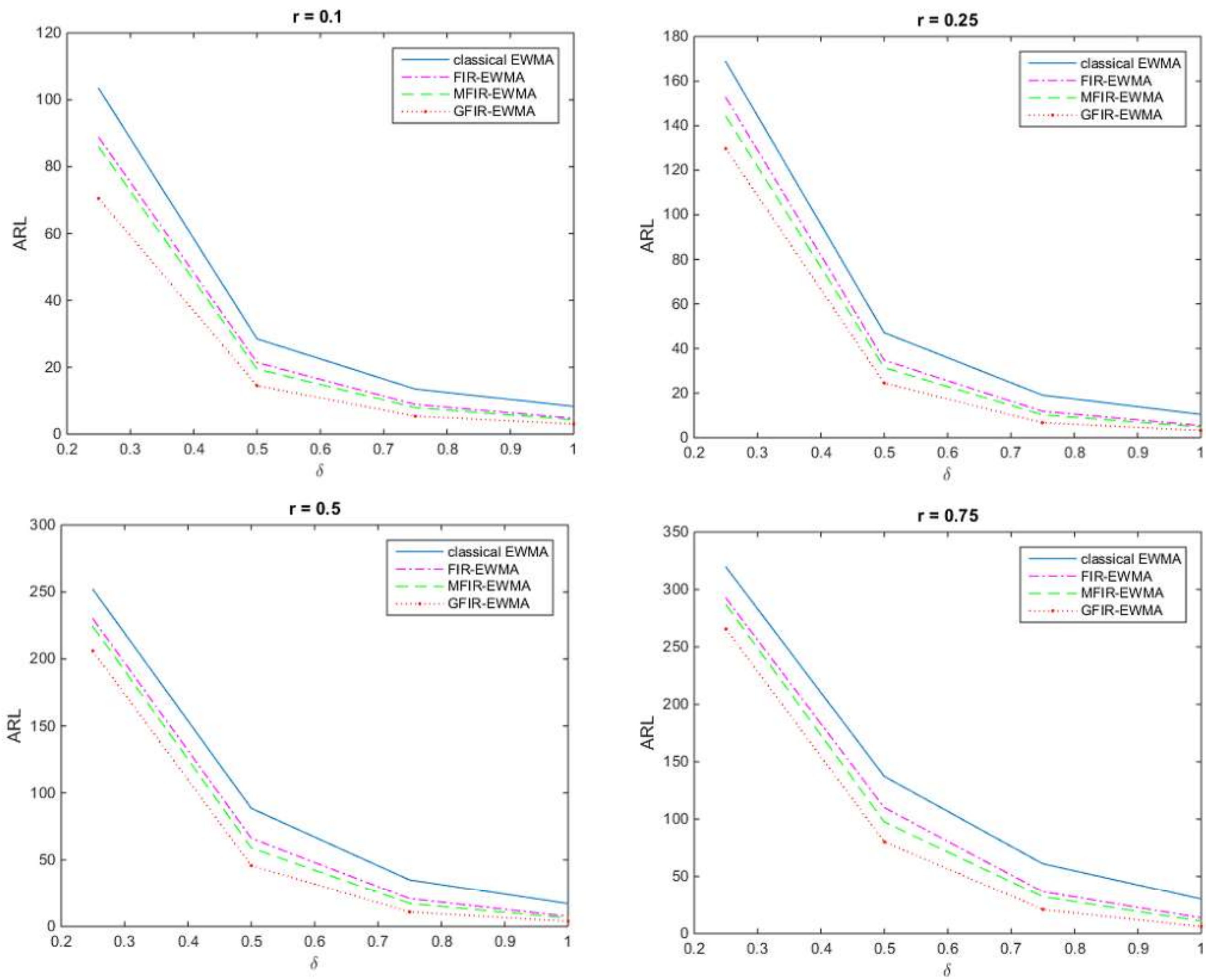

Figure 4.2: ARL comparison of classical EWMA, FIR-EWMA, MFIR-EWMA, and GFIR-EWMA for different values of r.

4.2.1 Results and Discussion

In this section, we discussed the time varying FIR features for control scheme. The ARL table for FIR-EWMA and MFIR-EWMA were reported in table (4.18) and table (4.19) respectively. We propose the generalized time varying FIR features for control charts. The ARL curves given in Figure (4.2) indicate that our proposed FIR feature can quickly detect problems in a process start up when the mean shifts are small. As the name suggests, generalized time varying FIR features for EWMA control scheme (GFIR-EWMA) gives the various values of headstart at different values of ϕ. In particular, at $\phi = 1/t$, it gives a 50% headstart. This is equivalent to the FIR feature suggested by Steiner[40].

4.3 Case Study via real life data set

In this section, we shall use a real-life data from a petroleum refinery laboratory to check the responsiveness of some of the control schemes discussed in this chapter to a mean shift. Di-Glycol Amine (DGA) is an Amine compound used in Petroleum refineries to eliminate sulfur compounds from petroleum gases by using a chemical process known as gas sweetening process. During this process, DGA eliminates sulfur species from the hydrocarbons and consequently, it decomposes into varieties of DGA degradation products. Its ability to remove more sulfur from the petroleum gases depends on the concentration of the remaining none decomposed DGA.

So, Chemical process experts send its used samples to the petroleum refining quality assurance laboratory for characterizing the used samples for variety of parameters including its used sample concentration. Based on the concentration level of the remaining used samples expressed as DGA wt%, the process experts either add make-up new or change the whole of it in the chemical process. The quality of its test result reported by the lab attendant paves way for process experts to take any reasonable action.

Potentiometric titration is an instrumental method widely used to find the concentration in the used DGA samples. The sensing part of this DGA lab analyzer is a pH probe which is calibrated by the lab attendant on regular basis. After calibration, the instrument performance is monitored using different types of usual control charts. The instructions given by the International Standard Procedure ASTM D6299[45] are strictly implemented. The data used to develop the charts are based on a quality control (QC) DGA sample prepared in house. This QC DGA standard (30.3 wt% Amine) is prepared by diluting 2, 2-Aminoethoxy ethanol (98% purity) with deionized water. The lab tests this Amine (DGA) control sample once per day.

The data used to monitor the purity of DGA analyzer performance are used here to plot some control charts we have discussed in this chapter. These charts are obtained using the decision parameters of the control procedures that give an $ARL_0 = 500$.

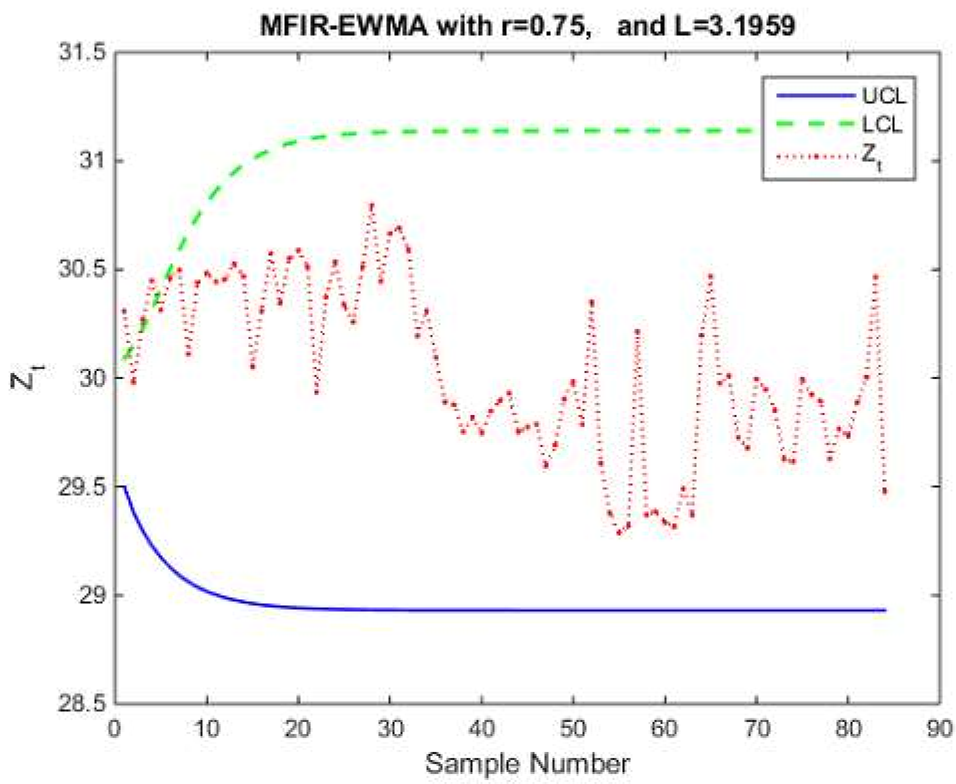

Figure 2.3(a): Control charts of the real data set discussed above

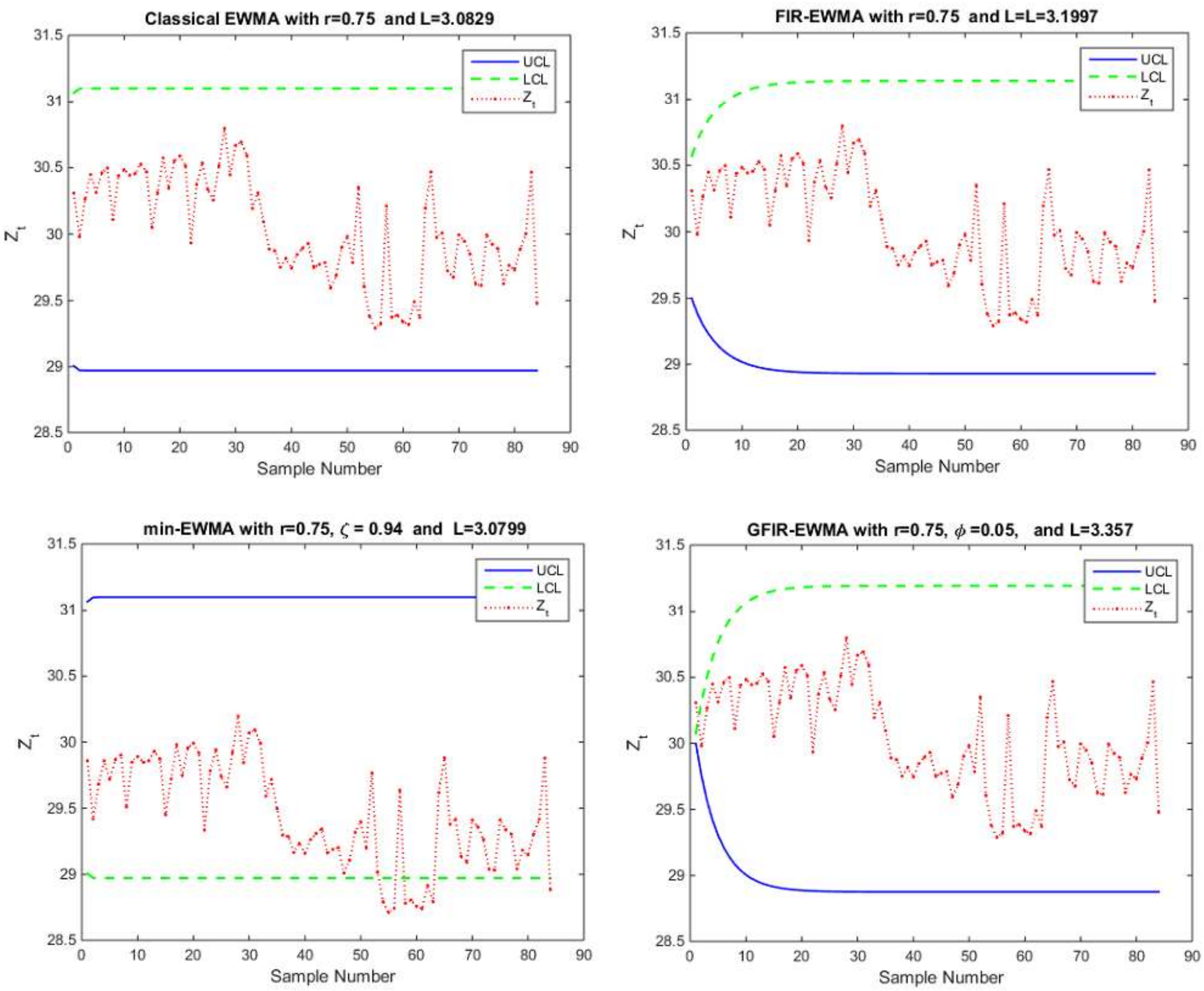

Figure 2.3(b): Control charts of the real data set discussed above

4.3.1 Results and Discussion

The control charts in Figure 2.3(a) and Figure 2.3(b) shows that the proposed min-EWMA has a great ability to detect shift when the weight parameter is large. We hereby recommend this procedure for monitoring processes where the underline mean shifts to detect are small or large.

4.4 The generalized control chart

Suppose that the individual random variables are normally distributed with mean (μ) and variance (σ^2). That is, $x_t \sim \mathcal{N}(\mu, \sigma^2)$. Let $\bar{x}_t$ be the mean of a random sample of n observations during the time interval t, then $\bar{x}_t \sim \mathcal{N}(\mu, \sigma^2)$. That is, $\bar{x}_t$ is normally distributed. All charts will be constructed for a standard normal variable

$$Z_t = \frac{\bar{x}_t - \mu}{\sigma/\sqrt{n}}, \quad Z_t \sim (0, 1). \tag{4.14}$$

The generalized control chart (Champ et al) [28] is based on the cumulative values

$$U_t = \max(-a_0,\ a_1 U_{t-1} + a_2 Z_t - a_3), \text{with}\, U_0 = a_4 \tag{4.15}$$

$$L_t = \min(-b_0,\ b_1 L_{t-1} + b_2 Z_t - b_3), \text{with}\, L_0 = b_4 \tag{4.16}$$

Where $a_i \geq 0 \,\forall\, i$ and $b_j \geq 0 \,\forall\, j$.

For the upper chart, the traditional Shewhart, Classical EWMA and CUSUM charts have the key parameters (a_0, a_1, a_2, a_3) as $(0, 0, 1, 0)$, $(0, 1, -r, r, 0)$ and $(0, 1, 1, k)$ respectively. The two-sided generalized procedure gives an out of control signal whenever $U_t \geq a_5$ or $L_t \leq -b_5$. The details can be found in [31].

4.4.1 EWMA Parameters

Following Lucas and Saccucci[32], the two-sided generalized control chart for EWMA are

$$U_t = \max(0, (1-r)U_{t-1} + rZ_t),\ U_0 = 0 \tag{4.17}$$

$$L_t = \min(0, (1-r)L_{t-1} + rZ_t),\ L_0 = 0 \tag{4.18}$$

where $r \in (0, 1]$.

4.4.2 CUSUM Parameters

Following Champ et al.[28], the two-sided generalized control procedure for CUSUM is given as follows

$$U_t = \max(0, U_{t-1} + Z_t - K),\ U_0 = 0 \tag{4.19}$$

$$L_t = \min(0, L_{t-1} + Z_t + K),\ L_0 = 0 \tag{4.20}$$

where $k \in \Re^+$.

An out of control signal is produced if $U_t \geq a_5$ or whenever $L_t \leq -b_5 = -a_5$. So, $a_5 = b_5$ when $n = \sigma = 1$. The details can be found in [30].

Based on (4.15), we define an equivalence control schemes

$$\mathcal{M}_t^1 = \min\{L_t, -U_t\} \tag{4.21}$$

$$\mathcal{M}_t^2 = \max\{-L_t, U_t\} \tag{4.22}$$

$$\mathcal{M}_t^3 = \max\{|L_t|, |U_t|\} \tag{4.23}$$

For $\mathcal{M}_t^1$, an out of control signal is produced if $\mathcal{M}_t^1 \leq -b_5$.

For $\mathcal{M}_t^2$, an out of control signal is produced if $\mathcal{M}_t^2 \geq a_5$.

For $\mathcal{M}_t^3$, an out of control signal is produced if $\mathcal{M}_t^3 \geq a_5$.

Table 4.21: The ARL values for $\mathcal{M}_t^1 EWMA$, $\mathcal{M}_t^2 EWMA$ and $\mathcal{M}_t^3 EWMA$.

$Mean$ $Shifts$ $\alpha\downarrow$	$Gen.EWMA$ $r=0.25$ $a_5=1.182$	$\mathcal{M}_t^1 EWMA$ $r=0.25$ $a_5=1.182$	$\mathcal{M}_t^2 EWMA$ $r=0.25$ $a_5=1.182$	$\mathcal{M}_t^3 EWMA$ $r=0.25$ $a_5=1.182$
0	501.7502	501.7502	501.7502	501.7502
0.25	196.017	196.017	196.017	196.017
0.5	56.9017	56.9017	56.9017	56.9017
0.75	23.1354	23.1354	23.1354	23.1354
1	12.2951	12.2951	12.2951	12.2951
1.25	8.0383	8.0383	8.0383	8.0383
1.5	5.8404	5.8404	5.8404	5.8404
1.75	4.585	4.585	4.585	4.585
2	3.832	3.832	3.832	3.832
2.25	3.2889	3.2889	3.2889	3.2889
2.5	2.8803	2.8803	2.8803	2.8803
2.75	2.5802	2.5802	2.5802	2.5802
3	2.3532	2.3532	2.3532	2.3532
3.25	2.1726	2.1726	2.1726	2.1726
3.5	2.0226	2.0226	2.0226	2.0226
3.75	1.9063	1.9063	1.9063	1.9063
4	1.7986	1.7986	1.7986	1.7986

On the basis of 10,000 Monte Carlo simulation from standard normal distribution, it is assumed that the in-control process is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 1$, that is $X_t \sim N(0,1)$. So, after a certain time t, a random shift δ occurs in the process mean. This shifts process mean from μ to a new mean level μ_1. That is, $\mu_1 = \mu + \delta\sigma$ or $\delta = \frac{|\mu_1 - \mu|}{\sigma_0}.$ Where δ represent the amount of shift n the process mean level. The ARL values for $\mathcal{M}_t^1 EWMA$, $\mathcal{M}_t^2 EWMA$ and $\mathcal{M}_t^3 EWMA$ for $r = 0.25$ have been calculated and reported in Table (4.21). The Table shows they have the same ARL values.

4.4.3 Results and Discussion

The generalized control charts have been proposed by Champ et al[28, 30, 31]. In this section, we discussed this control scheme briefly. Our contribution is that, we combined the upper and the lower chart together to get three different equivalent single but not one-sided control charts $\mathcal{M}_t^1$, $\mathcal{M}_t^2$, and $\mathcal{M}_t^2$. The ARL for the three equivalent control schemes at $r = 0.25$ were reported in Table (4.21). The ARL values validate the proposition that the three control charts are the same.

4.5 Concluding Remarks on Chapter 4

We conclude this chapter by pinpointing our contributions. Firstly, we propose min-EWMA control charts. The ARL values indicate that this scheme can quickly signal out of control in a process when the mean shifts of the process are small.

Secondly, we propose the generalized time varying FIR feature for control charts. The ARL values and graphical illustrations indicate that this scheme has the best performance among the other FIR features mentioned in this research work.

Thirdly, we provide a real life data as a case study for some control schemes mentioned in this chapter. The results and recommendations in this regard were discussed.

And lastly, We gave the three equivalent control schemes for generalized control charts. We present the ARL table of each scheme for some weighing parameter r for EWMA control chart.

CHAPTER 5

EWMA AR(1) PROCESSES AND OPTIMIZATION DESIGN

Some observations are usually independent and identically distributed (iid). However, in a practical sense, some are serially correlated as a first order autoregressive process, AR(1). Some researchers have recently shown an increasing interest in the formulation and analytical of non-Gaussian models for serially correlated data.

In this chapter, we shall discuss EWMA AR(1) processes for when the observation is exponentially white noise. Also, we shall discuss the MOP for EWMA AR(1) processes.

5.1 EWMA control chart for the Autoregressive process of order one AR(1) for the Exponential White Noise

The recursive equation for EWMA chart designed to detect any change in the mean of observed sequence of an AR(1) process with random variable X_t is given by

$$Z_t = rX_t + (1-r)Z_{t-1}, \qquad t = 1, 2, ... \tag{5.1}$$

with $Z_0 = u$ and $r \in (0, 1]$.

Where $Z_0 = u$ is an initial parameter often regarded as the target value.

In this study, we consider the sets of observations consisting of first order autoregressive AR(1) observations. The AR(1) process is defined as a solution of equation

$$X_t = \rho X_{t-1} + \zeta_t, \qquad t = 1, 2, ... \tag{5.2}$$

With $X_0 = v,\ (v,$ an initial value of $X_0)$

Where ζ_t is the white noise. We have assumed that the random variable ζ_t is the independent error term at time t following $\mathrm{Exp}(\alpha)$ (an Exponential distribution).

Let X be a random variable following an Exponential distribution. The probability density function of an Exponential distribution is given by

$$f(x;\alpha) = \begin{cases} \frac{1}{\alpha}e^{-\frac{x}{\alpha}} & x \geq 0 \\ \\ 0 & x < 0 \end{cases} \tag{5.3}$$

with mean $= \mathbf{E}(X) = \alpha$ and variance $= \mathbf{V}(X) = \alpha^2$. So, the standard deviation is α $(\alpha > 0)$. The variance of Z_t is derived as follows

$$\begin{aligned}
&\text{Var}\left(r\sum_{i=0}^{t-1}(1-r)^i Z_{t-1} + (1-r)^t Z_0\right) \\
&= \frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left[1-(1-r)^2 t\right] \\
&+2r^2\frac{\sigma_\zeta^2}{1-\rho^2}\sum_{i=0}^{t-2}\sum_{j=i+1}^{t-1}(1-r)^{i+j}\rho^{j-1} \\
&= \frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{1-r}\right)\left[1-(1-r)^2 t\right] \\
&+2r^2\frac{\sigma_\zeta^2}{1-\rho^2}\sum_{i=0}^{t-2}\left(\frac{1-r}{\rho}\right)\sum_{j=i+1}^{t-1}((1-r)\rho)^j \\
&= \frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left[1-(1-r)^{2t}\right] \\
&+2\frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r^2}{1-\rho(1-r)}\right) \\
&\left[\rho(1-r)\sum_{i=0}^{t-2}(1-r)^{2i} - (\rho(1-r))^t\sum_{i=0}^{t-2}\left(\frac{1-r}{\rho}\right)^i\right] \\
&= \frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left[1-(1-r)^{2t}\right] \\
&+2\frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left(\frac{\rho(1-r)}{1-\rho(1-r)}\right)\left[1-(1-r)^{2t-2}\right] \\
&-2\frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r^2}{1-\rho(1-r)}\right)\left(\frac{\rho^{t+1}(1-r)^t}{\rho+r-1}\right)\left[1-\left(\frac{1-r}{\rho}\right)^{t-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left(\frac{1+\rho(1-r)}{1-\rho(1-r)}\right)\left[1-(1-r)^{2t}\right] \\
&+2\frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left(\frac{\rho(1-r)}{1-\rho(1-r)}\right)\left[(1-r)^{2t}-(1-r)^{2t-2}\right] \\
&-2\frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r^2}{1-\rho(1-r)}\right)\left(\frac{\rho^2(1-r)^t}{\rho+r-1}\right)\left(\rho^{t-1}-(1-r)^{t-1}\right)
\end{aligned} \tag{5.4}
$$

For a large t, $\text{Var}(Z_t)$ is asymptotically equivalent to

$$
\frac{\sigma_\zeta^2}{1-\rho^2}\left(\frac{r}{2-r}\right)\left(\frac{1+\rho(1-r)}{1-\rho(1-r)}\right) \tag{5.5}
$$

Let L be the control limit multiplier and let μ be the mean for the process Z_t, the upper and lower control limits for a large t are given by

$$
UCL = h = \mu + L\sigma_\zeta\sqrt{\left(\frac{r}{2-r}\right)\left(\frac{1+\rho(1-r)}{(1-\rho^2)(1-\rho(1-r))}\right)} \tag{5.6}
$$

$$
LCL = 0 = \mu - L\sigma_\zeta\sqrt{\left(\frac{r}{2-r}\right)\left(\frac{1+\rho(1-r)}{(1-\rho^2)(1-\rho(1-r))}\right)} \tag{5.7}
$$

Where σ_ζ represent the standard deviation of a known probability distribution in context. The process will be out of control when $Z_t \geq h$. Thus, the alarm time to signal the out of control is given by

$$
\tau = \mathbf{inf}\{t > 0 : Z_t > h\} \tag{5.8}
$$

The measure ARL is commonly used to evaluate the performance of control charts at some discrete points of process shift within a specified range. The ARL of a charting structure is the number of samples to be taken before a false alarm is detected in the process. The performance can be evaluated by two values ARL_0 and ARL_1. ARL_0 is the expected number of samples before an out-of-control false alarm is detected when the process is at in-control state while ARL_1 is the expected number of samples before an out-of-control false alarm occurs when the process is shifted to an out-of-control state. A chart is considered to be more effective than other chart if it has a smaller ARL_1 (out-of-control ARL) values at more points Wu et al.[39].

Assume that $E_\theta(.)$ denote the expectation at time θ, where $\theta \leq \infty$, the average run length of the EWMA control chart for the given process is respectively given by

$$ARL_0 = E_\infty(\tau) = T \tag{5.9}$$

$$ARL_1 = E_1(\tau) \tag{5.10}$$

where T is given, usually large.

5.1.1 Explicit Formula for EWMA AR(1) Process for the Exponential White Noise

The performance of any control chart is measured by the average run length values (ARL). The ARL_0 is defined as the false alarm time τ before an in-control process signals to be out of control. A huge in-control ARL_0 is desired. On the other hand, (ARL_1) is the average run length when the process is out of control. The efficient of a control chart is accessed by the value of (ARL_1) or average delay time (AD). In contrast to (ARL_0), the minimum values of (ARL_1) is desired at the various mean shifts of the process.

The explicit formulas for the integral equation to find average run length values (ARL) for EWMA control chart for observations from AR(1) process with exponential white noise is given in [26]. Although in some real life cases, AR(1) process often follow normal distribution but in other situation, the AR(1) process may not follow a normal distribution as it may be rightly skewed.

When the initial value is u, let $L(u)$ denote the ARL of one-sided EWMA control chart. Since $\zeta_t \geq 0$, we then assume that the lower control limit (LCL)$= 0$ and upper limit is $h_{ucl} = h$ Putting $\zeta_1 = w$, $L(u)$ is given by the integral equation

$$L(u) = 1 + \int\limits_{0 \leq (1-r)u + r(\rho v + w) \leq h} L\left((1-r)u + rv + w\right) f(w) dw \tag{5.11}$$

By change of variables, we have

$$L(u) = 1 + \frac{1}{r} \int_{0}^{h} L(w) f\left(\frac{w - (1-r)u}{r} - \rho v\right) dw \tag{5.12}$$

Where L(u) is a Fredholm integral equation of the second kind. If we consider $w_t \sim Exp(\alpha)$, in (5.12), from Exponential distribution we have,

$$f(w) = \frac{1}{\alpha} e^{\frac{-w}{\alpha}},\ w \geq 0$$

$$f\left(\frac{w - (1-r)u}{r} - \rho v\right) = \frac{1}{\alpha} e^{\frac{-w}{r\alpha}} e^{\frac{1-r}{u} r\alpha} + \frac{\rho v}{\alpha}$$

So, (5.12) becomes

$$L(u) = 1 + \frac{1}{r\alpha} \int_{0}^{h} L(w) e^{\frac{-w}{r\alpha}} e^{\frac{1-r}{u} r\alpha} + \frac{\rho v}{\alpha} dw \tag{5.13}$$

Letting C(u) =

$$e^{\frac{1-r}{u}r\alpha} + \frac{\rho v}{\alpha}, \quad 0 \leq u \leq h,$$

we have

$$L(u) = 1 + \frac{C(u)}{r\alpha} \int_0^h L(w) e^{\frac{-w}{r\alpha}} dw, \quad 0 \leq u \leq h \tag{5.14}$$

putting

$$k = \int_0^h L(w) e^{\frac{-w}{r\alpha}} dw$$

then we get

$$L(u) = 1 + \frac{C(u)}{r\alpha} k \tag{5.15}$$

$$k = \int_0^h L(w) e^{\frac{-w}{r\alpha}} dw = \frac{-r\alpha(e^{\frac{-h}{r\alpha}} - 1)}{1 + (e^{\frac{-h}{\alpha}} - 1)\dfrac{e^{\frac{-\rho v}{\alpha}-1}}{r}} \tag{5.16}$$

now, by substituting (5.16) in (5.15), we finally arrive at

$$L(u) = ARL = 1 - \frac{r\mathrm{e}^{\frac{(1-r)u}{\alpha r}} \left(\mathrm{e}^{-\frac{h}{\alpha r}} - 1\right)}{r\mathrm{e}^{-\frac{\rho v}{\alpha}} + \mathrm{e}^{-\frac{h}{\alpha}} - 1}. \tag{5.17}$$

Thus, the equation (5.17) is the solution to the integral equation in (5.12).

5.2 Statistical design of control charts

Woodall [38] studied the statistical design of control charts. He gave the effect of an in-control region on the design of the Shewhart $\bar{X}$-chart and the two-sided CUSUM procedure. He recommended selecting the magnitude of the shift that is important to detect as a design criterion for control charts. He suggested designing control charts using three regions namely; In-control region, Indifferent region and Out-of-Control region.

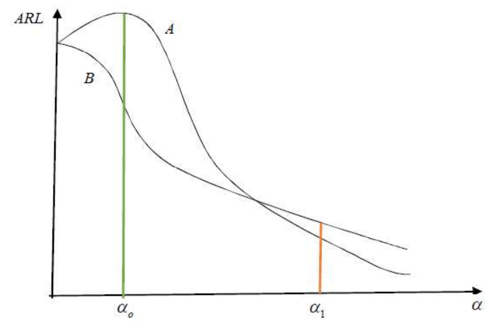

Figure 5.1: Woodall region (a)

Let α_0 be mean value of the process at in-control and let α_1 be the smallest shift in the mean considered important enough to be detected quickly

(1) **For the in-control region**, let $\alpha \in (0, \alpha_0]$ be the region that contains all the values of α (shifts) that is considered to be of no practical importance, thereby not requiring quick detection. In this region, the large value of ARL is desired.

(2) **For the out-of-control region**, let $\alpha \geq \alpha_1$ be the region where the chart detect a shift or disturbance in a process. It is regarded as a false alarm. The minimum values of ARL is desired here.

(2) **For the indifference region**, let $\alpha \in (\alpha_0, \alpha_1)$ be the region that corresponds to where the chart is indifferent if the shift is detected or not.

Woodall[38] used the following diagrams to explain these regions.

By considering the the two control schemes A and B, the ARL values are not considered over the indifference region. If the ARL curve of A is above that of B for when $\alpha \in (0, \alpha_0]$ and below that of B when $\alpha \geq \alpha_1$, then chart A is uniformly better than that of B. This is shown in figure 5.1.

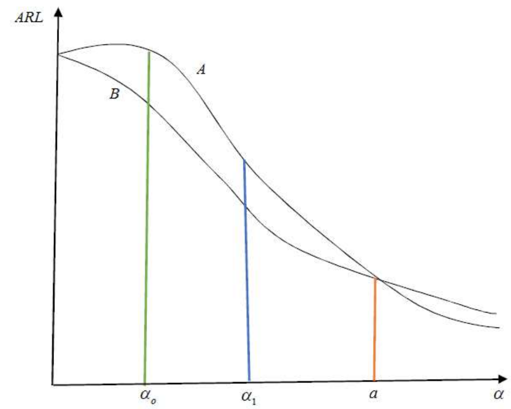

Figure 5.2: Woodall region (b)

In Figure 5.2, Chart A is better than B except on the interval $\alpha \in [\alpha_1, a)$. Figure 5.2 shows a common relationship between the ARL profiles of the Shewhart control chart (A) and the usual EWMA/CUSUM procedure (B).

5.3 Methodology for the multiobjective optimization design

The weighing parameter r for EWMA control scheme is an important decision parameter to monitor any process. The optimal design procedure would be to specify the desired in-control and out of control ARL with respect to the magnitudes of the mean shifts in the process[21]. In practice, to detect a small shift in the process mean, the values of r withing $0.05 \leq r \leq 0.25$ works fine[21]. Denote the mean shift here by α_1. In the case of a moderate process mean shift, the choice of r withing $0.25 \leq r \leq 0.5$ gives a better ARL performance. Denote the mean shift here by α_2.

And lastly, to detect a large process mean shift, the values of r withing $0.5 \leq r \leq 1$ works better since as $r \to 1$, EWMA chart becomes Shewhart $\bar{X}$ chart. This agrees with the charting properties of Shewhart $\bar{X}$ chart i.e. to detect large process mean shifts. Denote the mean shift here by α_3. The graphical interpretation is shown in Figure 5.2. Let f_1, f_2 and f_3 be the ARL functions corresponding to each mean region given above respectively. Our goal is to simultaneously minimize f_1, f_2 and f_3 that will give us a good ARL performance at each mean region. This leads to a conflicting objectives.

$$f_1(r, \rho, h) = 1 - \frac{r\mathrm{e}^{\frac{(1-r)u}{\alpha_1 r}}\left(\mathrm{e}^{-\frac{h}{\alpha_1 r}} - 1\right)}{r\mathrm{e}^{-\frac{\rho v}{\alpha_1}} + \mathrm{e}^{-\frac{h}{\alpha_1}} - 1} \tag{5.18}$$

$$f_2(r, \rho, h) = 1 - \frac{r\mathrm{e}^{\frac{(1-r)u}{\alpha_2 r}}\left(\mathrm{e}^{-\frac{h}{\alpha_2 r}} - 1\right)}{r\mathrm{e}^{-\frac{\rho v}{\alpha_2}} + \mathrm{e}^{-\frac{h}{\alpha_2}} - 1} \tag{5.19}$$

$$f_3(r, \rho, h) = 1 - \frac{r\mathrm{e}^{\frac{(1-r)u}{\alpha_3 r}}\left(\mathrm{e}^{-\frac{h}{\alpha_3 r}} - 1\right)}{r\mathrm{e}^{-\frac{\rho v}{\alpha_3}} + \mathrm{e}^{-\frac{h}{\alpha_3}} - 1} \tag{5.20}$$

Now, let $X_t, t \geq 0$ be a given process and let $\alpha_0 = 1$, be the mean at the process in-control. So, when there is a disturbance in the process, it shifts the mean from α_0 to α_3. Let $x = (r, \rho, h)$, following above discussions, we have the following MOP

$$\begin{aligned} &\text{minimize} \quad f(x) = [f_1(x), f_2(x), f_3(x)] \\ &\text{subject to} \\ &\qquad\qquad ARL_0 \geq \tau \\ &\qquad\qquad x \in \mathbf{S} \end{aligned}$$

Where u, v are initial values such that $0 \leq u < h,\ \tau = 370$ and

$$\mathbf{S} = \{(r, \rho, h) \in \Re^3 \mid \epsilon \leq r \leq 1, 0 \leq \rho \leq 1, h \geq 0, \epsilon > 0\}. \tag{5.21}$$

In practice, h is chosen to achieve a particular false alarm ARL_0. So, it is usually between the interval (0,10).

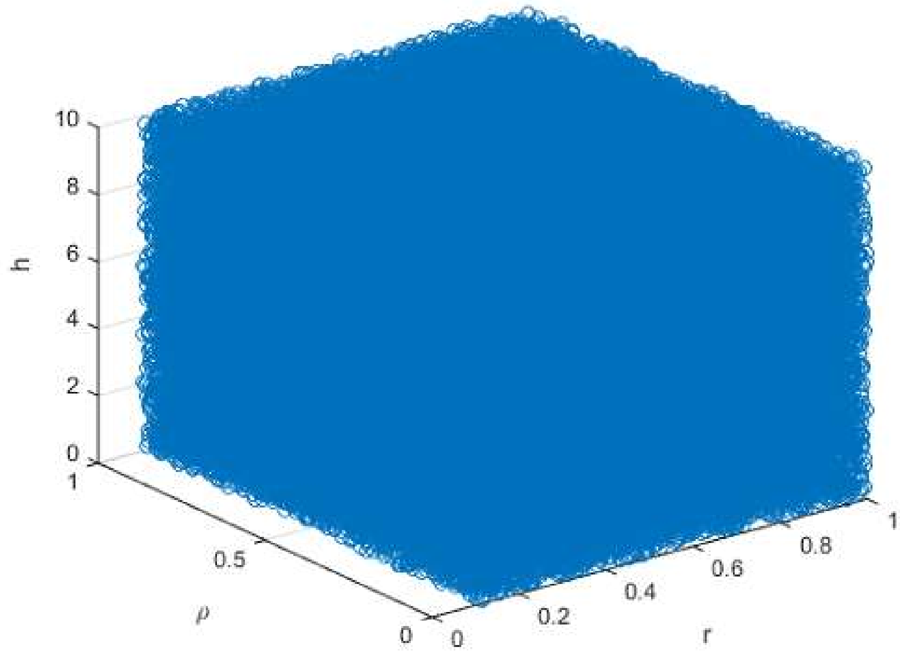

Figure 5.3: The feasible set of decision variables

Next, we check for the properties of the objective functions and the constraints.

- The constraint sets are in three dimensions. And by its definition, it is convex. The region showing the constraint sets is given in Figure 5.3.

- Convexity

The characterization for convexity of second order continuously partially differentiable functions is obtained by finding the Hessian matrix and show that it is either positive definite or positive semi-definite. This was investigated in the appendix $\mathbf{A_2}$. The result shows that f is not convex on convex set $\mathbf{S}$.

- Quasiconcavity

The characterization for quasiconcavity was discussed in chapter one. We used it to check for the quasi-concavity of f, and this was explicitly shown in appendix $\mathbf{A_1}$. The findings reveal that f is quasi-concave. This is summarized as a Theorem below

Theorem 5.1 *Let* $f : \mathbf{S} \rightarrow \Re^3$*, then the function*

$f_i : \mathbf{S} \rightarrow \Re$ *is quasiconcave for all* $(r, \rho, h) \in \mathbf{S}$.

The prove is explicitly given in Appendix $\mathbf{A_1}$.

Before we proceed to the MOP formulation, we shall briefly talk about Karush Kuhn Tucker conditions for f as this would add more value to this work. Due to the fact that f is highly nonlinear, we shall limit KKT discussion for the in-control process i.e $\alpha = 1$. Thus, for $\alpha = 1, \epsilon = 0.1, u = 0.1$ and $v = 0.1$, we have the optimization problem

$$\text{minimize} \quad 1 - \frac{e^{-\frac{0.1(r-1)}{r}} r\left(e^{-\frac{h}{r}} - 1\right)}{e^{-h} + e^{-0.1\rho} r - 1} \tag{5.22}$$

subject to

$$(r, \rho, h) \in \mathbf{S}$$

5.4 Karush Kuhn Tucker (KKT) First Order Condition

The KKT conditions were widely known as the Kuhn-Tucker conditions for many years, following the publication of a paper by Kuhn and Tucker describing them in 1951. Many years later it was observed that Karush had actually developed the same conditions much earlier in his Master's thesis in 1939 but Karush did not publish this fact from his master's work, and so it was unrecognized for decades[11]. The motivation of this story is for graduate students. Publish your work on time!

The KKT conditions are used to determine whether a point is or not a critical point of a given constrained nonlinear OP. They don't actually determine whether the point is a local optimal. It is just that it is a critical point and it could be a local maximal, a local minimal, or a saddle point[11]. The KKT conditions recognize two different possibilities for a local optimal point namely

(i) All the constraints are inactive at the local optimal point. In a situation like this, the gradient will be zero i.e. $\nabla f(x) = 0$ at the local optimal point.

(ii) At least one constraint is active at the local optimal point. In a situation like this, the gradient of the objective function is not zero i.e. $\nabla f(x) \neq$ at the local optimal point.

The KKT first order condition for (5.22) implies

$$\nabla f(r,\rho,h) + k^T\nabla(\sum_{j=1}^{n} g_j(r,\rho,h)) = 0 \tag{5.23}$$

$$k^T g_j(r,\rho,h) = 0 \tag{5.24}$$

Where $k_i's$ are Lagrangian multipliers. This gives us the system of equations

$$\begin{aligned}
&-\frac{e^{-\frac{0.1(r-1)}{r}}\left(\frac{0.1(r-1)}{r^2}-\frac{0.1}{r}\right)r\left(e^{-\frac{h}{r}}-1\right)}{e^{-h}+e^{-0.1\rho}r-1} \\
&-\frac{e^{-\frac{0.1(r-1)}{r}}\left(e^{-\frac{h}{r}}-1\right)}{e^{-h}+e^{-0.1\rho}r-1} \\
&+\frac{r\left(e^{-\frac{h}{r}}-1\right)e^{-0.1\rho-\frac{0.1(r-1)}{r}}}{\left(e^{-h}+e^{-0.1\rho}r-1\right)^2} \\
&-\frac{he^{-\frac{h}{r}-\frac{0.1(r-1)}{r}}}{r\left(e^{-h}+e^{-0.1\rho}r-1\right)}+\mathrm{k1}-\mathrm{k2}=0
\end{aligned} \tag{5.25}$$

$$-\frac{0.1r^2\left(e^{-\frac{h}{r}}-1\right)e^{-0.1\rho-\frac{0.1(r-1)}{r}}}{\left(e^{-h}+e^{-0.1\rho}r-1\right)^2}+\mathrm{k}3-\mathrm{k}4=0 \tag{5.26}$$

$$\frac{e^{-\frac{h}{r}-\frac{0.1(r-1)}{r}}}{e^{-h}+e^{-0.1\rho}r-1}-\frac{re^{-h-\frac{0.1(r-1)}{r}}\left(e^{-\frac{h}{r}}-1\right)}{\left(e^{-h}+e^{-0.1\rho}r-1\right)^2}+\mathrm{k}5-\mathrm{k}6=0 \tag{5.27}$$

$$\mathrm{k}_1(r-1)=0 \tag{5.28}$$

$$\mathrm{k}_2(0.1\;-r)=0 \tag{5.29}$$

$$\mathrm{k}_3(\rho-1)=0 \tag{5.30}$$

$$-\mathrm{k}_4\rho=0 \tag{5.31}$$

$$\mathrm{k}_5(h-10)=0 \tag{5.32}$$

$$\mathrm{k}_6(0.1\;-h)=0 \tag{5.33}$$

$$k_i\geq 0,\ \forall\, i=1,2,...,6.$$

In general, the KKT conditions are necessary to find an optimal point, but not necessarily sufficient. The KKT conditions are usually not solved directly in the case of highly nonlinear OP by software packages, Iterative successive approximation methods are often used. We can only use it to solve simple nonlinear OP usually with two or three constraints.

In particular, to solve Problem (5.25), we should be ready to solve systems of equations in 64 cases. This is as a consequence of the KKT optimality conditions. That is, 2^m cases, where m is the number of inequality constraints. So, in our case, there are six equations (5.25)-(5.33). Thereby, there are 2^6 cases and these equations are highly nonlinear and hence, there are difficulties in obtaining the KKT point for the OP (5.22).

In our attempt to solve this problem, we used Wolfram mathematical, Maple and Matlab softwares, but we were unable to obtain the KKT point. The usual messages were 'no feasible region found'. There after, we explored further options, we used the function handle fsolve together with some advanced optimization options in Matlab which later worked. At least, it gives us motivation to be optimistic at obtaining KKT point. Details on using fsolve can be found in [42].

Here, we only present the case where approximate KKT first order optimality condition is satisfied. The points are $r = 0.12, \rho = 0.50, h = 2.45, k_1 = 0.00, k_2 = 1.00, k_3 = 0.48, k_4 = 0.49, k_5 = 0.00, k_6 = 0.00$ and the optimum value for f is 7.66.

Thank God, a very powerful algorithm (Genetic Algorithm)has been developed to solve highly nonlinear optimization problems[42]. It is a robust optimization software as it can solve any highly nonlinear or non-smooth OP which might be difficult to be solved using standard optimization method. We shall discuss Genetic algorithm briefly in subsequent sections.

5.5 Scalarization

We discussed scalarization method of solving MOP in Chapter one. The Weighted sum method for the MOP in (5.21) is given by

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{3} w_i f_i(x) \\
\text{subject to} \quad & \\
& ARL_0 \geq \tau \\
& x \in \mathbf{S}
\end{aligned}
\tag{5.34}
$$

where $w_i > 0$, $\forall\, i = 1, 2, ..., 3$.

5.6 Optimum Search

5.6.1 Algorithm

An algorithm is a series of information consisting a finite set of well-defined instructions to solve a given problem or a task. An algorithm starts in an initial state and stops in a designate end-state. Turing and Church (1936) formalized the concept of an algorithm. The goal of algorithm in optimization is to find a solution either specific values for the decision variables or one specific decision alternatives with minimal or maximal evaluation value. Simplex method is a classical method to solve an optimization problem whose objectives and the constraints are linear. In the case of non-linear, there are several methods that could be used to optimize the optimization problem. In particular, if the problem is quadratic, then one can apply steepest descent or Newton's method to find the minimum or the maximum point of the function. However, some optimization problems are not well suited for standard optimization algorithms including problems in which the objective function is discontinuous, non-differentiable, stochastic or highly nonlinear. In these types of problems, **genetic algorithm** is the most appropriate to optimize the decision variables under consideration.

5.6.2 Genetic algorithm

The genetic algorithm is a method for solving optimization problems that based on natural selection. It is the process that drives biological evolution. The search for the global optimum value in an optimization problem is accomplished when an initial population (which is known as generation) of individuals passes to a new population. That is, next generation through the application of genetic operators. It uses three main types of rules at each step to make the next generation from the current population. The three rules are:

- Selection rules: It selects the individuals called the parents, that contribute to the population at the next generation.

- Crossover rules: This combine two parents to form children for the next generation.

- Mutation rules: This apply random changes to individual parents to form children.

In the original population, each individual represents a possible optimal solution to the optimization problem.

The objective function is used to provide an assessment of how individuals have performed in the problem domain. This is also known as fitness function. It indicates the fitness of that individual with respect to the other individual of the population. This raw measure of fitness is usually only used as an intermediate state in determining the relative performance of individuals in a genetic algorithm. So, its correct definition allows for a better use to the algorithm since finding the global optimum is only being monitored by the fitness function.

Due to the probabilistic nature of Genetic algorithm, each single run might give different optimal values. So, we incorporated it with Monte Carlo simulation. The supplied initial parameters are $u = 0.01, v = 0.1$ and $ARL_0 = 370$. The simulation was made 5000 times at a constant numbers of generations in Matlab.

We stored the values of the decision variables and the objective functions in each run. This was used to plot the feasible and the objective space of this problem in three dimensions and two dimensions. These are given Figure 5.4 and Figure 5.5 respectively. By virtue of our discussions in Chapter 3, the ordering cone of the objective space is pointed and consequently, the optimal values obtained are strongly Pareto optimal.

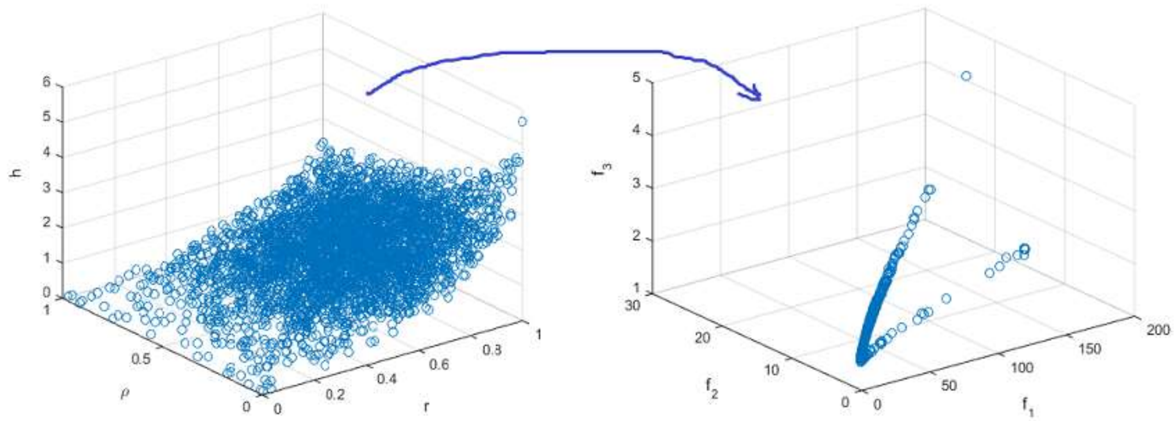

Figure 5.4: Feasible set and the Objective space in three dimensions.

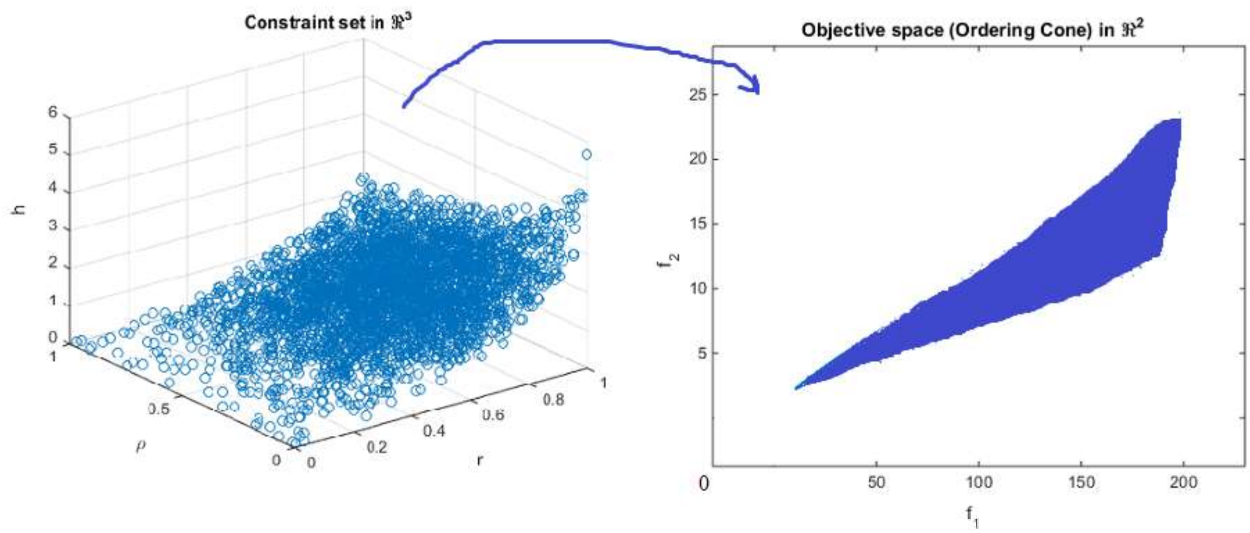

Figure 5.5: Feasible set and the Objective space in three and two dimensions respectively.

The Pareto optimal values obtained are $f_1 = 17.33, f_2 = 3.12, f_3 = 1.57.$ and $r = 0.79, \rho = 0.57h = 1.37.$

And the optimal values for the Weighted sum Scalarization method at $w_1 = 1, w_2 = 1$ and $w_3 = 1$ are $f = 27.89, r = 0.89, \rho = 0.58$ and $h = 1.83.$

5.6.3 Performance Comparison

Here, we compare the obtained optimal points with the EWMA and MEWMA points obtained by Aparisi[19]. He obtained $r = 0.85$ and $L = 3.09$ for the univariate and multivariate EWMA control scheme. It should be noted that h is a control limit. Its major role is to control the limit for which observations are being monitored. Its an important parameter for setting the false alarm i.e. $ARL_0(\tau)$. The table below summarizes our results when the false alarm ARL_0 is fixed to 370.

Table 5.1: The performance comparison of ARL and $ARL_{Aparisi}$ when $ARL_0 = 370$

α (Mean shifts) $\rightarrow$	$\tau \downarrow$	1.1	1.8	4
ARL $(r = 0.79, \rho = 0.57, h = 1.37)$	370	17.33	3.12	1.57
$ARL_{Aparisi}$ $(r = 0.85, \rho = 0.58, h = 1.61)$	370	19.80	3.41	1.64

Next, we present a case study using random numbers generated from Exponential distribution. We consider a process $X_t, t \geq 0$ whose in-control mean is 0.1. So, after some time, there is a disturbance in the process which shifts the mean to points 0.18,0.22 and 0.4. This corresponds to small, moderate and large mean shift respectively. Our goal is to check the control procedure which has the ability of detecting the shifts at the earlier time interval. Five thousands random variables is generated using the mean shifts specifications above. Using the control parameters given in Table 5.1 and following Equations 5.1 and 5.2 we obtain the following control charts.

Note that different random variables will definitely give different charts at each run. So, we provide the seed Matlab code that give those charts in the Appendix 6.4 . Discussions on Table 5.1, Chart 5.6 and Chart 5.7 are given in the results and discussion in Subsection 5.6.4.

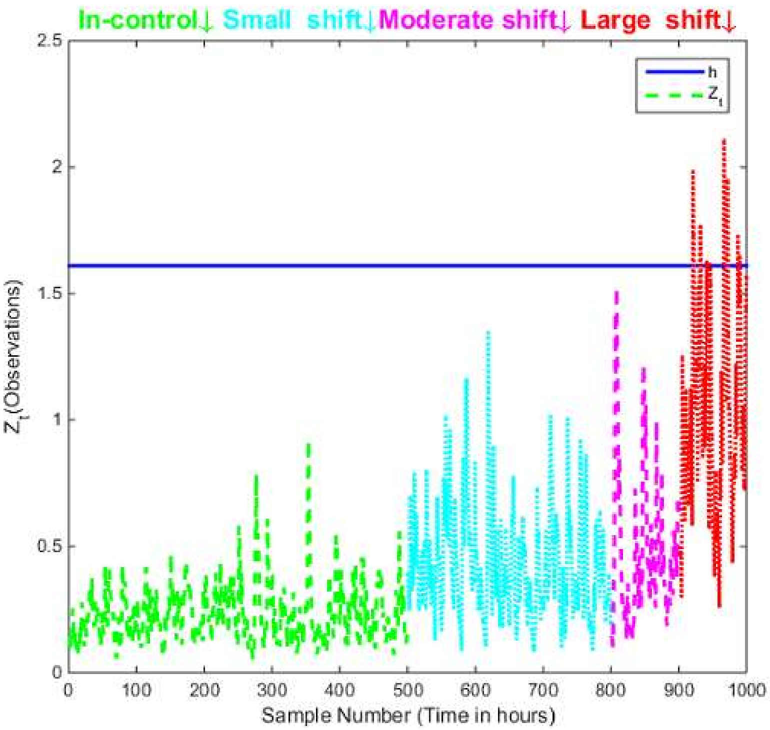

Figure 5.6: Control chart of the case study using specifications of $ARL_{Aparisi}$ when $ARL_0 = 370$.

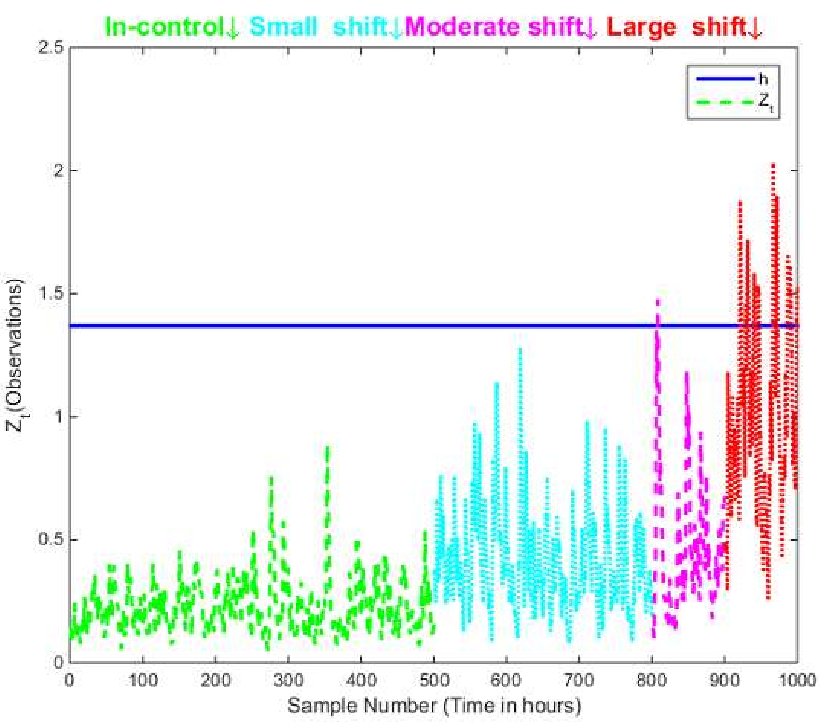

Figure 5.7: Control chart of the case study using the optimal points of ARL. values.

5.6.4 Results and Discussion

Each parameter or decision variable has its own practical interpretation. For r, it is a weighing parameter and its influence on any control chart was discussed in Chapter 5. The optimal points obtained above is a trade off point for the three mean shifts regions. And, the industrial exploration to a refinery laboratory in Saudi Arabia during the course of this work revealed that a sufficient large value of r is mostly used 'ISO' in the industries. Otherwise, most of the processes will always go out-of-control if at all nothing has gone wrong in the process.

Cisar[43] gave the following interpretations for autoregressive coefficients ρ.

- When $\rho \in [0, 0.2]$, it means no correlation or insignificant correlation.
- When $\rho \in (0.2, 0.4]$, it corresponds to low correlation.
- When $\rho \in (0.4, 0.6]$, it indicates a moderate correlation.
- When $\rho \in (0.6, 0.8]$, it means a significant correlation.
- When $\rho \in (0.8, 1]$, it means a high correlation.

So, the autoregressive coefficient value obtained is within the moderate region. This works fine for any process. Note that h is a control limit. Its value depends on the values of r, ρ and ARL_0 (false alarm). Based on these, we recommend these optimal points for monitoring industrial processes. Especially, the ones that involve autocorrelation.

Furthermore, For the ARL values given in Table 5.1, the ARL_1 value of the ARL is smaller than that of the ARL_1 of $ARL_{Aparisi}$, hence the proposed MOP approach performs better. Similarly, in Figure 5.7, the out of control was detected at sample number 800 (time), while in Figure 5.6, the out of control was detected at the sample number 900 (time). This means that our proposed scheme has a greater detecting ability than the optimal point obtained by Aparisi[19]. So, in both cases, the Proposed MOP approach performs better.

5.7 Conclusive Remarks

In this Chapter, we investigated the properties of the objective functions and the constraint sets. We showed that the objective functions are quasiconcave. The ARL response to its decision parameters at the various level of process mean shifts was used to obtain several conflicting objectives. We present the MOP design for this control scheme. The objectives were combined as a single objective by Weighted sum method. The Pareto optimal values were calculated and reported. The ARL was calculated and reported and the case study using Exponential distribution random variables was presented. And lastly, we present results and discussions on these optimal points.

CHAPTER 6

CONCLUSION AND RECOMMENDATION

6.1 Conclusion on Chapter 3

In Chapter 3, we discussed some concepts of optimization and in particular, MOP. We mentioned the existence theorems for minimal or maximal element of a set. The notions on optimality for MOP problems was equally discussed. Weighted sum scalarization approach for MOP was discussed.

6.2 Conclusion on Chapter 4

In Chapter 4, we propose min-EWMA control charts. The ARL values indicate that this scheme can quickly signal out of control in a process when the mean shifts of the process are small.

Secondly, we propose the generalized time varying FIR feature for control charts. The ARL values and graphical illustrations indicated that this scheme has the best performance among the other FIR features mentioned in this research work.

Thirdly, we provide a real life data as a case study for some control schemes mentioned in this Chapter. The results and recommendations in this regard were discussed.

And lastly, We gave the three equivalent control schemes for generalized control charts. We present the ARL table of each scheme for some weighing parameter r for EWMA control chart.

6.3 Conclusion on Chapter 5

In Chapter 5, we investigated the properties of the objective functions and the constraint sets. We showed that the objective functions are quasiconcave. The ARL response to its decision parameters at the various level of process mean shifts was used to obtain several conflicting objectives. We present the MOP design for this control scheme. The objectives were combined as a single objective by Weighted sum method. The Pareto optimal values were calculated and reported. The ARL was calculated and reported and the case study using Exponential distribution random variables was presented. And lastly, we present results and discussions on these optimal points.

6.4 Recommendation

Our recommendation for future work is that, the ARL functions for multivariate EWMA control scheme and for some other control schemes should be investigated. This will bring about ease and quick detection of mean shifts since simulation takes more time.

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Appendix

A1 : Quasiconcativity property of f

Given that

$$f = 1 - \frac{r\left(e^{-\frac{h}{\alpha r}} - 1\right)e^{-\frac{(r-1)u}{\alpha r}}}{e^{-\frac{h}{\alpha}} + re^{-\frac{\rho v}{\alpha}} - 1} \tag{6.1}$$

The Boredered Hessian for three variables is define as

$$\mathbf{D}_k(f, r, \rho, h) = det\begin{pmatrix} 0 & \frac{\partial f}{\partial_r} & \frac{\partial f}{\partial_\rho} & \frac{\partial f}{\partial_h} \\ \frac{\partial f}{\partial_r} & \frac{\partial^2 f}{\partial_r^2} & \frac{\partial^2 f}{\partial_r \partial_\rho} & \frac{\partial^2 f}{\partial_r \partial_h} \\ \frac{\partial f}{\partial_\rho} & \frac{\partial^2 f}{\partial_\rho \partial_r} & \frac{\partial^2 f}{\partial_\rho^2} & \frac{\partial^2 f}{\partial_\rho \partial_h} \\ \frac{\partial f}{\partial_h} & \frac{\partial^2 f}{\partial_h \partial_r} & \frac{\partial^2 f}{\partial_h \partial_\rho} & \frac{\partial^2 f}{\partial_h^2} \end{pmatrix}, k = 1, 2, 3. \tag{6.2}$$

We say that f is quasiconcave on a solid (non-empty interior) convex set Ω if

$$(-1)^k \mathbf{D}_k(f, r, \rho, h) \geq 0, \quad k = 1, 2, 3.$$

and quasiconvex if

$$(-1)^k \mathbf{D}_k(f, r, \rho, h) \leq 0, \quad k = 1, 2, 3.$$

The derivatives are given below

$$\begin{aligned}
\frac{\partial f}{\partial_r} = &-\frac{r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}}\left(\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)}{e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1} \\
&-\frac{\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}}}{e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1} \\
&+\frac{r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2} \\
&-\frac{he^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)}
\end{aligned}$$

$$\frac{\partial f}{\partial_\rho} = -\frac{r^2 v \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{(r-1)u}{\alpha r} - \frac{\rho v}{\alpha}}}{\alpha \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^2}$$

$$\frac{\partial f}{\partial_h} = \frac{e^{-\frac{h}{\alpha r} - \frac{(r-1)u}{\alpha r}}}{\alpha \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)} - \frac{r \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{h}{\alpha} - \frac{(r-1)u}{\alpha r}}}{\alpha \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^2}$$

$$\frac{\partial^2 f}{\partial_r^2} = -\frac{r \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{(r-1)u}{\alpha r}} \left(\frac{(r-1)u}{\alpha r^2} - \frac{u}{\alpha r}\right)^2}{e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1} - \frac{2 \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{(r-1)u}{\alpha r}} \left(\frac{(r-1)u}{\alpha r^2} - \frac{u}{\alpha r}\right)}{e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1} + \frac{2r \left(e^{-\frac{h}{\alpha r}} - 1\right) \left(\frac{(r-1)u}{\alpha r^2} - \frac{u}{\alpha r}\right) e^{-\frac{(r-1)u}{\alpha r} - \frac{\rho v}{\alpha}}}{\left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^2}$$

$$-\frac{h\left(\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)e^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)}$$

$$-\frac{he^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}\left(\frac{h}{\alpha r^2}+\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)}{\alpha r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)}$$

$$-\frac{r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}}\left(\frac{2u}{\alpha r^2}-\frac{2(r-1)u}{\alpha r^3}\right)}{e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1}$$

$$+\frac{2\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$-\frac{2r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{2\rho v}{\alpha}}}{\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^3}$$

$$+\frac{2he^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$\frac{\partial^2 f}{\partial_r\partial_\rho}=-\frac{r^2v\left(e^{-\frac{h}{\alpha r}}-1\right)\left(\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$+\frac{2r^2v\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{2\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^3}$$

$$-\frac{hve^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$-\frac{2rv\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$\frac{\partial^2 f}{\partial_r\partial_h}=\frac{he^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha^2r^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)}$$

$$-\frac{r\left(e^{-\frac{h}{\alpha r}}-1\right)\left(\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$+\frac{\left(\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)e^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)}$$

$$-\frac{he^{-\frac{h}{\alpha}-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha^2r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$-\frac{\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$+\frac{2r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^{3}}-\frac{e^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^{2}}$$

$$\frac{\partial^2 f}{\partial_\rho \partial_r}=-\frac{r^2v\left(e^{-\frac{h}{\alpha r}}-1\right)\left(\frac{(r-1)u}{\alpha r^2}-\frac{u}{\alpha r}\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^{2}}+\frac{2r^2v\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{2\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^{3}}-\frac{hve^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^{2}}-\frac{2rv\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^{2}}$$

$$\frac{\partial^2 f}{\partial_\rho^2} = \frac{r^2 v^2 \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{(r-1)u}{\alpha r} - \frac{\rho v}{\alpha}}}{\alpha^2 \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^2} - \frac{2 r^3 v^2 \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{(r-1)u}{\alpha r} - \frac{2\rho v}{\alpha}}}{\alpha^2 \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^3}$$

$$\frac{\partial^2 f}{\partial_\rho \partial_h} = \frac{r v e^{-\frac{h}{\alpha r} - \frac{(r-1)u}{\alpha r} - \frac{\rho v}{\alpha}}}{\alpha^2 \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^2} - \frac{2 r^2 v \left(e^{-\frac{h}{\alpha r}} - 1\right) e^{-\frac{h}{\alpha} - \frac{(r-1)u}{\alpha r} - \frac{\rho v}{\alpha}}}{\alpha^2 \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^3}$$

$$\frac{\partial^2 f}{\partial_h \partial_r} = -\frac{r \left(e^{-\frac{h}{\alpha r}} - 1\right) \left(\frac{(r-1)u}{\alpha r^2} - \frac{u}{\alpha r}\right) e^{-\frac{h}{\alpha} - \frac{(r-1)u}{\alpha r}}}{\alpha \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)^2} + \frac{e^{-\frac{h}{\alpha r} - \frac{(r-1)u}{\alpha r}} \left(\frac{h}{\alpha r^2} + \frac{(r-1)u}{\alpha r^2} - \frac{u}{\alpha r}\right)}{\alpha \left(e^{-\frac{h}{\alpha}} + r e^{-\frac{\rho v}{\alpha}} - 1\right)}$$

$$-\frac{he^{-\frac{h}{\alpha}-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha^2 r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$-\frac{\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$+\frac{2r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^3}$$

$$-\frac{e^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$\frac{\partial^2 f}{\partial_h\partial_\rho}=\frac{rve^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$-\frac{2r^2v\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}-\frac{\rho v}{\alpha}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^3}$$

$$\frac{\partial^2 f}{\partial_h^2} = \frac{r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{h}{\alpha}-\frac{(r-1)u}{\alpha r}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2} - \frac{2r\left(e^{-\frac{h}{\alpha r}}-1\right)e^{-\frac{2h}{\alpha}-\frac{(r-1)u}{\alpha r}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^3}$$
$$-\frac{e^{-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha^2 r\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)}$$
$$+\frac{2e^{-\frac{h}{\alpha}-\frac{h}{\alpha r}-\frac{(r-1)u}{\alpha r}}}{\alpha^2\left(e^{-\frac{h}{\alpha}}+re^{-\frac{\rho v}{\alpha}}-1\right)^2}$$

$$\mathbf{D}_1 = det\begin{pmatrix} 0 & \dfrac{\partial f}{\partial_r} \\ \dfrac{\partial f}{\partial_r} & \dfrac{\partial^2 f}{\partial_r^2} \end{pmatrix}$$

$$= -\frac{e^{\frac{2(r-1)(h-u)+2\rho r v}{\alpha r}}\left(re^{h/\alpha}\left(u\left(e^{\frac{h}{\alpha r}}-1\right)+h\right)-\left(e^{h/\alpha}-1\right)e^{\frac{\rho v}{\alpha}}\left(\left(e^{\frac{h}{\alpha r}}-1\right)(u-\alpha r)+h\right)\right)^2}{\alpha^2 r^2\left(e^{h/\alpha}\left(r-e^{\frac{\rho v}{\alpha}}\right)+e^{\frac{\rho v}{\alpha}}\right)^4} \tag{6.3}$$

$$\Rightarrow (-1)^1\mathbf{D}_1 \geq 0 \quad \forall \quad r,\rho,h \in \mathbf{S}$$

$$\mathbf{D}_2 = det \begin{pmatrix} 0 & \dfrac{\partial f}{\partial_r} & \dfrac{\partial f}{\partial_\rho} \\ \dfrac{\partial f}{\partial_r} & \dfrac{\partial^2 f}{\partial_r^2} & \dfrac{\partial^2 f}{\partial_r \partial_\rho} \\ \dfrac{\partial f}{\partial_\rho} & \dfrac{\partial^2 f}{\partial_\rho \partial_r} & \dfrac{\partial^2 f}{\partial_\rho^2} \end{pmatrix}$$

$$= \frac{1}{\alpha^4 \left(e^{h/\alpha}\left(r - e^{\frac{\rho v}{\alpha}}\right) + e^{\frac{\rho v}{\alpha}}\right)^6}$$

$$\left(e^{\frac{h}{\alpha r}} - 1\right) \exp\left(\frac{h(4r-3) + 3(-ru + \rho r v + u)}{\alpha r}\right)$$

$$\Big(2\alpha r^3 e^{\frac{2h}{\alpha}}(h-u) - 2\alpha r^3 u e^{\frac{2h(r+1)}{\alpha r}} +$$

$$(u - \alpha r)^2 e^{\frac{2(h+\rho r v)}{\alpha r}} + (u - \alpha r)^2 e^{\frac{2(hr+h+\rho r v)}{\alpha r}} - 2(u-\alpha r)^2 e^{\frac{h(r+2)+2\rho r v}{\alpha r}} -$$

$$2(\alpha r - u)(h + \alpha r - u)e^{\frac{h+2\rho r v}{\alpha r}} + 4e^{\frac{hr+h+2\rho r v}{\alpha r}}$$

$$(\alpha r - u)(h + \alpha r - u) - 2e^{\frac{2hr+h+2\rho r v}{\alpha r}}$$

$$(\alpha r - u)(h + \alpha r - u) + e^{\frac{2\rho v}{\alpha}}(h + \alpha r - u)^2 +$$

$$(h+\alpha r-u)^2 e^{\frac{2(h+\rho v)}{\alpha}} - 2(h+\alpha r-u)^2 e^{\frac{h+2\rho v}{\alpha}}$$

$$r^2 + e^{\frac{2hr+h}{\alpha r}} \left(h^2 - 2\alpha hr + 4\alpha ru\right) +$$

$$r\left(h^2 - 4\alpha hr + 2hu - 4\alpha^2 r^2 + 8\alpha ru - 2u^2\right) e^{\frac{hr+h+\rho rv}{\alpha r}} +$$

$$r\left(4\alpha r(h-u) + (h-u)^2 + 2\alpha^2 r^2\right) e^{\frac{h+\rho v}{\alpha}} - re^{\frac{2h+\rho v}{\alpha}}$$

$$\left(4\alpha r(h-u) + (h-u)^2 + 2\alpha^2 r^2\right) - re^{\frac{2h(r+1)+\rho rv}{\alpha r}}$$

$$2\alpha^2 r^2 - 4\alpha ru + u^2\Big) + r\left(2\alpha^2 r^2 - 4\alpha ru + u^2\right) e^{\frac{h(r+2)+\rho rv}{\alpha r}} +$$

$$r\left(-h^2 + 4\alpha r(h-2u) - 2hu + 4\alpha^2 r^2 + 2u^2\right) e^{\frac{2hr+h+\rho rv}{\alpha r}}\Big) \qquad (6.4)$$

$$\Rightarrow (-1)^2 \mathbf{D}_2 \geq 0 \quad \forall \quad r, \rho, h \in \Omega.$$

$$\mathbf{D}_3 = det \begin{pmatrix} 0 & \dfrac{\partial f}{\partial_r} & \dfrac{\partial f}{\partial_\rho} & \dfrac{\partial f}{\partial_h} \\ \dfrac{\partial f}{\partial_r} & \dfrac{\partial^2 f}{\partial_r^2} & \dfrac{\partial^2 f}{\partial_r \partial_\rho} & \dfrac{\partial^2 f}{\partial_r \partial_h} \\ \dfrac{\partial f}{\partial_\rho} & \dfrac{\partial^2 f}{\partial_\rho \partial_r} & \dfrac{\partial^2 f}{\partial_\rho^2} & \dfrac{\partial^2 f}{\partial_\rho \partial_h} \\ \dfrac{\partial f}{\partial_h} & \dfrac{\partial^2 f}{\partial_h \partial_r} & \dfrac{\partial^2 f}{\partial_h \partial_\rho} & \dfrac{\partial^2 f}{\partial_h^2} \end{pmatrix}$$

$$= -\left(v \exp\left(\frac{h(5r-4)+4(-ru+\rho rv+u)}{\alpha r}\right)\left(r^5\left(e^{\frac{h}{\alpha r}}-1\right)\left(-e^{\frac{\rho v}{\alpha}}\right)\right.\right.$$

$$\alpha v\left(r\left(-e^{h/\alpha}\right)+e^{\frac{h+\rho v}{\alpha}}-e^{\frac{\rho v}{\alpha}}\right)^3$$

$$\left(re^{h/\alpha}(h-u)+rue^{\frac{h(r+1)}{\alpha r}}+(u-\alpha r)e^{\frac{h+\rho rv}{\alpha r}}\right.$$

$$(\alpha r-u)+e^{\frac{hr+h+\rho rv}{\alpha r}}+e^{\frac{\rho v}{\alpha}}(h+\alpha r-u)-$$

$$\left.(h+\alpha r-u)e^{\frac{h+\rho v}{\alpha}}\right)\left(\alpha re^{\frac{2h+\rho v}{\alpha}}-\alpha(r-1)re^{\frac{\rho v}{\alpha}}\right.$$

$$\alpha r^2\left(-e^{\frac{2h}{\alpha}}\right)-\alpha(r-1)r^2e^{h/\alpha}+(u-\alpha r)e^{\frac{2hr+h+\rho rv}{\alpha r}}$$

$$r+e^{\frac{2hr+h}{\alpha r}}(\alpha r-u)+r^2(\alpha r-u)e^{\frac{h(r+2)+\rho rv}{\alpha r}}+$$

$$re^{\frac{h(r+1)}{\alpha r}}+(-hr+\alpha(r-1)r+u)e^{\frac{h+\rho rv}{\alpha r}}$$

$$-hr+\alpha(r-1)r+u-e^{\frac{hr+h+\rho rv}{\alpha r}}\left(h(r-1)r-2\left(r^2-1\right)u\right.$$

$$-hr+\alpha(r-1)r+u-e^{\frac{hr+h+\rho rv}{\alpha r}}\left(h(r-1)r-2\left(r^2-1\right)u\right.$$

$$r^5\left(e^{\frac{h}{\alpha r}}-1\right)e^{\frac{\rho v}{\alpha}}\left(r\left(-e^{h/\alpha}\right)+e^{\frac{h+\rho v}{\alpha}}-e^{\frac{\rho v}{\alpha}}\right)^3$$

$$v\left(re^{h/\alpha}+re^{\frac{h+\rho rv}{\alpha r}}-e^{\frac{h+\rho v}{\alpha}}+(r-1)\left(-e^{\frac{\rho v}{\alpha}}\right)\right)$$

$$\alpha\left(2\alpha r^3ue^{\frac{h(r+2)}{\alpha r}}-\alpha r^2(\alpha r-2u)e^{\frac{2h+\rho rv}{\alpha r}}+\alpha r(h-u)e^{\frac{2h+\rho v}{\alpha}}\right.$$

$$
\begin{aligned}
&\alpha \Big(2\alpha r^3 u e^{\frac{h(r+2)}{\alpha r}} - \alpha r^2(\alpha r - 2u)e^{\frac{2h+\rho r v}{\alpha r}} + \alpha r(h-u)e^{\frac{2h+\rho v}{\alpha}} \\
&\alpha r - 2ru + +u) + \alpha r \left(2h(r-1) + \alpha r^2 - 2(r-1)u\right) e^{\frac{h+\rho v}{\alpha}} - \alpha r e^{\frac{\rho v}{\alpha}} \\
&\left(h(2r-1) + \alpha r^2 - 2ru + u\right) + r\left(-e^{\frac{2hr+h}{\alpha r}}\right) + (h(u-\alpha r) + \alpha r u)e^{\frac{2hr+h+\rho r v}{\alpha r}} \\
&h(u-\alpha r) + \alpha r(\alpha r + u) + \left(h^2 r - 2h(\alpha(r-1)r + u) - 2\alpha r\left(\alpha r^2 - 2ru + u\right)\right) \\
&e^{\frac{hr+h+\rho r v}{\alpha r}} + r e^{\frac{h(r+1)}{\alpha r}} \left(-h^2 r + h(\alpha r(2r-1) + u) + \alpha r(\alpha r - 4ru + u)\right) + \\
&\left(-h^2 r + h(\alpha r(2r-1) + u) + \alpha r\left(2\alpha r^2 - 4ru + u\right)\right) e^{\frac{h+\rho r v}{\alpha r}}\Big) - \\
&\alpha r^7 v e^{h/\alpha}\left(e^{\frac{h}{\alpha r}} - 1\right)^2 \left(r e^{h/\alpha} - e^{\frac{h+\rho v}{\alpha}} + e^{\frac{\rho v}{\alpha}}\right)^3 \\
&\Big(2\alpha r^3 u e^{\frac{h(r+2)+\rho r v}{\alpha r}} + 2\alpha r^2 u e^{\frac{2hr+h}{\alpha r}} - 3\alpha^2 r^2 e^{\frac{2h+\rho v}{\alpha}} + 2\alpha^2 r e^{\frac{2(h+\rho v)}{\alpha}} \\
&\alpha^2 r^3 + e^{\frac{2h}{\alpha}} - \alpha r^2(\alpha r - 2u)e^{\frac{2(h+\rho r v)}{\alpha r}} + \alpha r(2\alpha r - 5u)e^{\frac{2hr+h+\rho r v}{\alpha r}} \\
&\alpha(2\alpha r - 3u)\left(-e^{\frac{2hr+h+2\rho r v}{\alpha r}}\right) + \alpha r^2(2\alpha r - 3u)e^{\frac{h(r+2)+2\rho r v}{\alpha r}} - e^{\frac{h+\rho v}{\alpha}} r^2 \alpha(h(2r-1) \\
&\alpha(2r-3) - 2ru + +u) - \alpha r e^{\frac{2\rho v}{\alpha}}\left(h(2r-1) + \alpha\left(r^2 + r - 2\right) - 2ru + u\right) + \\
&\alpha r\left(h(3r-1) + \alpha\left(2r^2 + r - 4\right) - 3ru + u\right) e^{\frac{h+2\rho v}{\alpha}} + \\
&\alpha\left(\alpha r\left(2r^2 + r - 2\right) + \left(-4r^2 + r + 3\right)u\right)\right) + e^{\frac{hr+h+2\rho r v}{\alpha r}} h^2 r - h(\alpha r(3r-2) + u) \\
&-\alpha\left(\alpha r\left(4r^2 + r - 4\right) + \left(-6r^2 + r + 6\right)u\right)\Big)\Big)\Big) \Big/ \alpha^7 r^6 \left(r e^{h/\alpha} - e^{\frac{h+\rho v}{\alpha}} + e^{\frac{\rho v}{\alpha}}\right)^{11} \quad (6.5)
\end{aligned}
$$

$$\Rightarrow (-1)^3\mathbf{D}_3 \geq 0 \quad \forall \quad r, \rho, h \in \Omega.$$

hence, f is quasiconcave on Ω

A2 : Convexity

The Hessian matrix is given by

$$\mathbf{H}_k(f, r, \rho, h) = det \begin{pmatrix} \dfrac{\partial^2 f}{\partial_r^2} & \dfrac{\partial^2 f}{\partial_r \partial_\rho} & \dfrac{\partial^2 f}{\partial_r \partial_h} \\ \dfrac{\partial^2 f}{\partial_\rho \partial_r} & \dfrac{\partial^2 f}{\partial_\rho^2} & \dfrac{\partial^2 f}{\partial_\rho \partial_h} \\ \dfrac{\partial^2 f}{\partial_h \partial_r} & \dfrac{\partial^2 f}{\partial_h \partial_\rho} & \dfrac{\partial^2 f}{\partial_h^2} \end{pmatrix}, k = 1, 2, 3. \qquad (6.6)$$

Direct verification as above shows that, f is not convex.

A2 : Matlab code for the Charts 5.7 and 5.6 .

```
clear all

close all

clc

%% THIS IS THESIS CASE STUDIES USING EXP RAN-
DOM NUMBERS BY GANIYU AJIBADE AYODELE

r= 0.85 ; rhho=0.58; h= 1.61;

rng(125344,'twister')

incontrol = exprnd(0.1,[1 500]);

smallshift = exprnd(0.18, [1 300]);

moderateshift = exprnd(0.22,[1 100]);

bigshift = exprnd(0.4, [1 100]);

outofcontrol = [smallshift,
moderateshift, bigshift];
```

```
observation= [incontrol, outofcontrol];

hh = ones(size(observation)) * h;

X=ones(1,length(observation));

Z= ones(1,length(observation));

d=1:length(observation);

i=1; Z(1)= u; X(1)= v;

for i = 2:length(observation)

X(i)= rhho*X(i-1) + observation(i);

Z(i)= r*X(i) + (1-r)*Z(i-1);

end plot(d(1:1:length(incontrol)),

Z(1:1:length(incontrol)),'g–'...

,   d(1:1:length(incontrol)),   hh(1:1:length(incontrol)),'b-'...

,'LineWidth',2);
```

hold on plot(d(length(incontrol)+1:1:end),

Z(length(incontrol)+1:1:end),'r–'... ,

d(length(incontrol)+1:1:end),

hh(length(incontrol)+1:1:end),'b-'... ,'LineWidth',2);

$xlabel('SampleNumber(Timeinhours)')$

$ylabel('Z_t(Observations)')$

$title('Regionmarkedwith : greenindicateincontrol\&$

$redindicateoutofcontrol')$

$legend('h',' Z_t')$

Vitae

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- St. Andrew's Primary School, Isinbode, Ekiti State, Nigeria, First School Leaving Certificate, (1991-1997).

Outcomes/Publications

1. Ganiyu A.A, *Dynamic response to Multi-objective optimization problems*, Nigeria Association of Mathematics Physics (NAMPS, 2015). Conference paper.

2. Ganiyu A.A, Alshahrani M. Riaz M, *Evolutionary Multiobjective Optimization for EWMA AR(1) processes when the observation is trend exponential*, Society for Industrial and Applied Mathematics. To be presented by July, 2016 at SIAM annual meeting, Boston, USA.

3. *Enhanced universal control charts.* To be submitted.

4. *A Multiobjective Optimization design for EWMA AR(1) processes.* To be submitted.

5. *On the time varying FIR features for Statistic process control.* To be submitted.